Logical Constraints

Given the optimization problem:

$$\textbf{Max} \qquad z = f(x_1,\ x_2,\ \ldots,\ x_n)$$

$$\text{s.t.} \qquad g_1(x_1, x_2, \ldots, x_n) \geq 0 \qquad (1)$$

$$g_2(x_1, x_2, \ldots, x_n) \geq 0 \qquad (2)$$

$$\ldots$$

Only a given "combination" of the constraints must be imposed.

Only One Constraint

Only one of the two constraints (1) and (2) must be imposed.

* **Let L_1 and L_2 be the "inferior limits" of functions g_1 and g_2, respectively (f. i., - *M*, with *M* very large positive number).**

* **Let t_1 and t_2 be two *binary* variables such that:**
$t_k = 0$ if constraint (*k*) is imposed, $t_k = 1$ otherwise ($k = 1, 2$);

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t_1 L_1 \qquad (1a)$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t_2 L_2 \qquad (2a)$$

and impose the additional constraints:

$$t_1 + t_2 = 1, \quad \text{with } t_1, t_2 \text{ binary variables } (t_1, t_2 \in \{0,1\})$$

Not Both Constraints

Or only (1), or only (2), or none ot the two constraints must be imposed.

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t_1 L_1 \qquad (1a)$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t_2 L_2 \qquad (2a)$$

and impose the additional constraints:

$$t_1 + t_2 \geq 1,$$

with $t_1, t_2 \in \{0,1\}$

Not Only One Constraint

Or both (1) and (2), or none ot the two constraints must be imposed.

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t L_1 \qquad \textbf{(1b)}$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t L_2 \qquad \textbf{(2b)}$$

with $t \in \{0,1\}$

At Least One Constraint

Or only (1), or only (2), or both constraints must be imposed.

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t_1 L_1 \qquad \text{(1a)}$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t_2 L_2 \qquad \text{(2a)}$$

with $t_1 + t_2 \leq 1,$

$$t_1, \; t_2 \in \{0,1\}$$

Minimum Production Lots

- **Production of n items.**
- M_j **minimum "lot" of item j ($j = 1,\ldots, n$) (minimum quantity to be produced).**
- x_j **= quantity of item j to be produced.**
- **At least a quantity M_j of item j is produced ($x_j \geq M_j$) or item j is not produced ($x_j = 0$), $j = 1,\ldots, n$.**

*** Let t_j be a *binary* variable such that:**

$$t_j = 0 \text{ if } (x_j = 0), \quad \text{and} \quad t_j = 1 \text{ if } (x_j \geq M_j).$$

- **Impose the following constraints ($j = 1,\ldots, n$) :**

$$x_j \geq M_j\, t_j \; ; \quad x_j \leq G\, t_j \; ; \quad t_j \in \{0,1\}$$

with G very large positive number.

Minimum Production Lots

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- **Impose the following constraints ($j = 1,\ldots, n$) :**

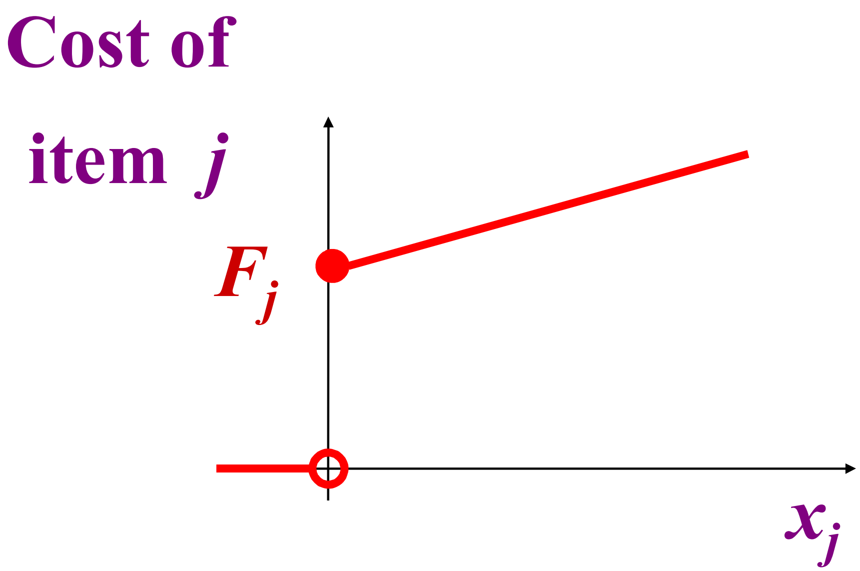
$$x_j \geq M_j t_j ; \quad x_j \leq G t_j ; \quad t_j \in \{0,1\}$$

with G very large positive number.

- **If $t_j = 0$: $x_j \leq 0$ and $x_j \geq 0$, hence: $x_j = 0$;**
- **If $t_j = 1$: $x_j \leq G$ and $x_j \geq M_j$, hence: $x_j \geq M_j$**

Fixed Production Cost

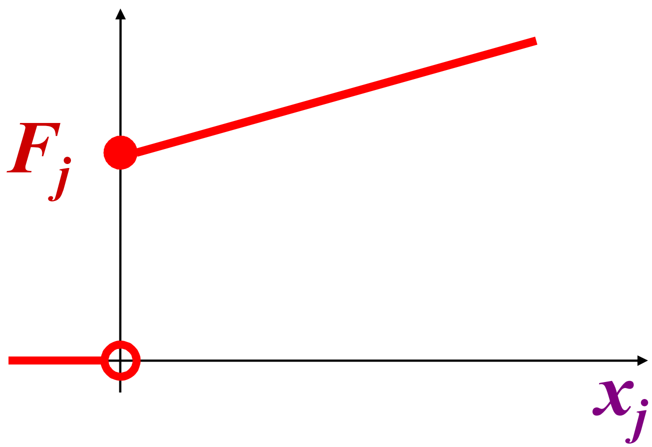
- Production Problem concerning n items.
- The cost of item j ($j = 1,\ldots, n$) is given by:
  - 0, if item j is not produced ($x_j = 0$)
  - $F_j + p_j x_j$ if item j is produced ($x_j > 0$)



- F_j = fixed cost (machine)
- p_j = unit production cost

- discontinuity at the origin (not algebraic function)

Fixed Production Cost (2)



$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases}$$

$(j = 1, \ldots, n)$

Min $\Sigma_{j=1,n} (F_j y_j + p_j x_j)$

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \\ y &\in \{0,1\} \end{aligned}$$

Logical constraints (*if* …)

Variables x and y must be connected through linear constraints

Fixed Production Cost (3)

$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases} \Rightarrow x_j \le M y_j \quad (\boldsymbol{j = 1, \ldots, n})$$

con $M >> 1$ ($\cong +\infty$)

- **Satisfaction of the constraint:**
  - **if $x_j > 0$ y_j must be = 1 ($x_j \le$ M)**
  - **if $x_j = 0$ y_j can be = 0 or 1 ($0 \le 0$ or $0 \le M$)**

or

  - **if $y_j = 0$ x_j must be = 0 ($x_j \le$ 0) (since $x_j \ge$ 0)**
  - **if $y_j = 1$ x_j can be = 0 or > 0 ($x_j \le M$)**

- **The constraint imposes only a part of the logical relation**

Fixed Production Cost (4)

$$\text{Min } \sum_{j=1,n} (F_j y_j + p_j x_j)$$

$$Ax \geq b$$

$$x_j \leq My_j$$

$$x_j \geq 0, \quad y_j \in \{0, 1\} \qquad (j = 1, \ldots, n)$$

MILP Model (Mixed Integer Linear Programming)

- **It is not necessary to impose also the other part of the logical relation:**
- **a solution with $x_j = 0$ and $y_j = 1$ cannot be optimal (an alternative feasible solution with a smaller cost exists)**

Fixed Production Cost (4)

Min $\Sigma_{j=1,n} (F_j\, y_j + p_j\, x_j)$

$$Ax \geq b$$

$$x_j \leq My_j$$

$$x_j \geq 0, \quad y_j \in \{0, 1\} \qquad (j = 1, \ldots, n)$$

MILP Model (Mixed Integer Linear Programming)

- **It is not necessary to impose also the other part of the logical relation:**
- **a solution with $x_j = 0$ and $y_j = 1$ cannot be optimal (an alternative feasible solution with a smaller cost exists: $x_j = 0, \; y_j = 0$)**

Discrete Variables

* Variable x must have a value among k (with $k > 1$) given different values:

$$x \in S = \{s_1, s_2, \ldots, s_k\}$$

* Let t_h ($h = 1, 2, \ldots, k$) be a *binary* variable such that:

$$t_h = 1 \text{ if } (x = s_h), \text{ and } t_h = 0 \text{ if } (x \neq s_h).$$

* $x = \sum_{h=1,k} s_h \, t_h$

with $\sum_{h=1,k} t_h = 1$

and $t_h \in \{0, 1\}$ ($h = 1, \ldots, k$)

Discrete Variables (2)

* **Example**

$$x \in S = \{0.2, 0.4, \ldots, 2.0\} \quad (k = 10)$$

- $t_h \in \{0, 1\}$ with $h = 1, 2, \ldots, 10$
- $x = 0.2\, t_1 + 0.4\, t_2 + 0.6\, t_3 + \ldots + 2.0\, t_1$

with

$$\Sigma_{h=1,10}\; t_h = 1$$

* **Alternative technique:**

$$x = 0.2\, y \quad \text{with } y \geq 1, \quad y \leq 10, \quad y \;\; \textit{integer}$$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

* x integer variable with $x \geq 0$, $x \leq k$

* Introduce $(k + 1)$ binary variables t_h , with $h = 0, \ldots, k$
($t_h = 1$ if $x = h$, and $t_h = 0$ otherwise)

$$x = \Sigma_{h = 0, k} \; h \, t_h$$

$$\Sigma_{h = 0, k} \; t_h = 1 \quad \text{with } t_h \in \{0, 1\} \; h = 0, \ldots, k$$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- x integer variable with $x \geq 0$, $x \leq k$

Alternative technique (binary expression of an integer):

introduce q binary variables t_h , with $h = 1, \ldots, q$

$$x = \Sigma_{h=1,q} \, 2^{h-1} t_h = t_1 + 2\,t_2 + 4\,t_3 + \ldots + 2^{q-1} t_q$$

$$\Sigma_{h=1,q} \, 2^{h-1} t_h \leq x \qquad t_h \in \{0, 1\} \; h = 1, \ldots, q \quad (**)$$

$$q = \lceil z \rceil \qquad \text{with} \quad z = log_2\,(k+1)$$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- ***Example:*** x **integer variable with** $0 \leq x \leq 27$

- $z = log_2\ (27 + 1)$, $q = \lceil z \rceil = 5$

- $x = t_1 + 2\,t_2 + 4\,t_3 + 8\,t_4 + 16\,t_5$

- **We must impose the constraint:**

$$t_1 + 2\,t_2 + 4\,t_3 + 8\,t_4 + 16\,t_5 \leq 27 \quad (**)$$

(for $t_1 = t_2 = t_3 = t_4 = t_5 = 1$**, we should have:** $x = 31$**)**

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- ***Example, Alternative Transformation:***
- x **integer variable with** $0 \leq x \leq 27, \quad q = 5$

12 = 16 – (31 – 27)

- $x = t_1 + 2\,t_2 + 4\,t_3 + 8\,t_4 + 12\,t_5$
- **We have not to impose the constraint:**

$$t_1 + 2\,t_2 + 4\,t_3 + 8\,t_4 + 12\,t_5 \leq 27 \quad (**)$$

(for $t_1 = t_2 = t_3 = t_4 = t_5 = 1$, we have: $x = 27$)

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- x integer variable with $x \geq b$, $x \leq k$ with $b \neq 0$

introduce q binary variables t_h, with $h = 1, \ldots, q$

$$x = \Sigma_{h=1,q}\ 2^{h-1} t_h + b \quad , q = \lceil z \rceil, z = log_2\ (k - b + 1)$$

$$\Sigma_{h=1,q}\ 2^{h-1} t_h \leq x - b \qquad t_h \in \{0, 1\}\ h = 1, \ldots, q \quad (**)$$

Example: $5 \leq x \leq 8$, $q = 2$, $x = 5 + t_1 + 2\, t_2$

Knapsack Problem (KP01)

Given:

n **items,**

P_j **"profit" of item *j*,** $j = 1, \ldots, n$ $(P_j > 0)$,

W_j **"weight" of item *j*,** $j = 1, \ldots, n$ $(W_j > 0)$,

one **container ("knapsack") with "capacity"** C:

determine a subset of the *n* items so as to maximize the global profit, and such that the global weight is not larger than the knapsack capacity *C*.

KP01 is NP-Hard

Knapsack Problem (KP01)

Determine a subset of items so as to maximize the global profit, and such that the global weight is not larger than the knapsack capacity C.

We assume:

$n \geq 2$

$P_j > 0, \qquad j = 1, \ldots, n$

$W_j > 0, \; W_j \leq C, \qquad j = 1, \ldots, n$

$\sum_{j=1,n} W_j > C$

Polynomial Problems

A *Polynomial Problem* can be solved *in the worst case* through *an algorithm* whose *computing time* grows according to a *polynomial function* of the *size* of the problem.

- **Examples:**

  given an array A_j of n elements ($j = 1, \ldots, n$):

- **Determine the *minimum value* of the n elements: $O(n)$ time.**

- ***Sort* the n elements according to non-decreasing (or non-increasing) values: $O(n \log n)$ time.**

Particular Cases of KP01

a) Constant Profits:

$$P_j = K \qquad (j = 1, \ldots, n)$$

1) *Sort* the *n* items according to non-decreasing values of the weights W_j : ($O(n \log n)$ time);

2) Insert the items until the first item *s* is found such that:

$$\sum_{j=1,s} W_j > C \qquad \text{(items } s, s+1, \ldots, n \text{ are not inserted):}$$

$O(n)$ time.

Global computing time: $O(n \log n) + O(n)$: $O(n \log n)$

Particular Cases of KP01

b) Constant Weights:

$$W_j = K \qquad (j = 1, \ldots, n)$$

1) *Sort* the n items according to non-increasing values of the profits P_j : ($O(n \log n)$ time);

2) Insert the items until the first item s is found such that:

$$\sum_{j=1,s} W_j > C \qquad \text{(items } s, s+1, \ldots, n \text{ are not inserted):}$$

$O(n)$ time.

Global computing time: $O(n \log n) + O(n)$: $O(n \log n)$

Mathematical Model KP01

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is inserted in the knapsack} \\ 0 & \text{otherwise} \end{cases} \qquad (j = 1, \ldots, n)$$

max $$\sum_{j=1,n} P_j \, x_j$$

$$\sum_{j=1,n} W_j \, x_j \le C$$

$$x_j \in \{0, 1\} \qquad (j = 1, \ldots, n)$$

or $$0 \le x_j \le 1 \quad \textit{integer} \quad (j = 1, \ldots, n)$$

ILP Model (Binary Linear Programming Model)

There exist items j with $P_j < 0$ and $W_j < 0$

Let $N = \{j\!: P_j > 0$ and $W_j > 0,\ j = 1, \ldots, n\}$;

Let $R = \{j\!: P_j < 0$ and $W_j < 0,\ j = 1, \ldots, n\}$.

Set $y_j = x_j,\ P'_j = P_j,\ W'_j = W_j \qquad j \in N$

Set $y_j = 1 - x_j,\ P'_j = -P_j,\ W'_j = -W_j\ (x_j = 1 - y_j) \quad j \in R$

$$Z = \sum_{j=1,n} P_j x_j = \sum_{j \in N} P_j y_j + \sum_{j \in R} P_j (1 - y_j) =$$

$$\sum_{j \in N} P'_j y_j + \sum_{j \in R} P'_j y_j + \sum_{j \in R} P_j = \sum_{j=1,n} P'_j y_j + a$$

where $a = \sum_{j \in R} P_j$

There exist items j with $P_j < 0$ and $W_j < 0$

Let $N = \{j: P_j > 0 \text{ and } W_j > 0, j = 1, \ldots, n\}$;

Let $R = \{j: P_j < 0 \text{ and } W_j < 0, j = 1, \ldots, n\}$.

Set $y_j = x_j, \; P'_j = P_j, \; W'_j = W_j \qquad j \in N$

Set $y_j = 1 - x_j, \; P'_j = -P_j, \; W'_j = -W_j \; (x_j = 1 - y_j) \quad j \in R$

$$Z = \sum_{j=1,n} W_j x_j = \sum_{j=1,n} P'_j y_j + a \quad \text{where } a = \sum_{j \in R} P_j$$

$$\sum_{j=1,n} W_j x_j = \sum_{j=1,n} W'_j y_j + b \quad \text{where } b = \sum_{j \in R} W_j$$

There exist items j with $P_j < 0$ and $W_j < 0$

$$\max \quad \sum_{j=1,n} P'_j\, y_j + a$$

$$\sum_{j=1,n} W'_j\, y_j \le C' \quad \text{(where } C' = C - b\text{)}$$

$$y_j \in \{0, 1\} \qquad (j = 1, \ldots, n)$$

$$x_j = y_j \quad j \in N; \qquad x_j = 1 - y_j \quad j \in R$$