Logical Constraints

Given the optimization problem:

$$\textbf{Max} \qquad z = f(x_1, x_2, \ldots, x_n)$$

s.t.

$$g_1(x_1, x_2, \ldots, x_n) \geq 0 \qquad (1)$$

$$g_2(x_1, x_2, \ldots, x_n) \geq 0 \qquad (2)$$

$$\ldots$$

Only a given "combination" of the constraints must be imposed.

Only One Constraint

Only one of the two constraints (1) and (2) must be imposed.

- **Let L_1 and L_2 be the "inferior limits" of functions g_1 and g_2, respectively (f. i., - *M*, with *M* very large positive number).**

*** Let t_1 and t_2 be two *binary* variables such that:**
$t_k = 0$ if constraint (*k*) is imposed, $t_k = 1$ otherwise ($k = 1, 2$);

*** Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t_1 L_1 \qquad (1a)$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t_2 L_2 \qquad (2a)$$

and impose the additional constraint:

$$t_1 + t_2 = 1, \quad \text{with } t_1, t_2 \text{ binary variables } (t_1, t_2 \in \{0,1\})$$

Not Both Constraints

Or only (1), or only (2), or none ot the two constraints must be imposed.

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t_1 L_1 \qquad \text{(1a)}$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t_2 L_2 \qquad \text{(2a)}$$

and impose the additional constraint:

$$t_1 + t_2 \geq 1,$$

with $t_1, t_2 \in \{0,1\}$

Not Only One Constraint

Or both (1) and (2), or none ot the two constraints must be imposed.

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t L_1 \qquad \text{(1b)}$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t L_2 \qquad \text{(2b)}$$

with $t \in \{0,1\}$

At Least One Constraint

Or only (1), or only (2), or both constraints must be imposed.

* **Replace constraints (1) and (2) with:**

$$g_1(x_1, x_2, \ldots, x_n) \geq t_1 L_1 \qquad \text{(1a)}$$

$$g_2(x_1, x_2, \ldots, x_n) \geq t_2 L_2 \qquad \text{(2a)}$$

and impose the additional constraint:

$$t_1 + t_2 \leq 1,$$

with $t_1, t_2 \in \{0,1\}$

* **Combinations of 3 or more constraints**

Minimum Production Lots

- **Production of n items.**
- **M_j minimum "lot" of item j ($j = 1,\ldots, n$) (minimum quantity to be produced).**
- **x_j = quantity of item j to be produced.**
- **At least a quantity M_j of item j is produced ($x_j \geq M_j$) or item j is not produced ($x_j = 0$), $j = 1,\ldots, n$.**

*** Let t_j be a *binary* variable such that:**

$$t_j = 0 \text{ if } (x_j = 0), \quad \text{and} \quad t_j = 1 \text{ if } (x_j \geq M_j).$$

- **Impose the following constraints ($j = 1,\ldots, n$) :**

$$x_j \geq M_j \, t_j \, ; \quad x_j \leq G \, t_j \, ; \quad t_j \in \{0,1\}$$

with G very large positive number.

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- Impose the following constraints ($j = 1,\ldots, n$) :

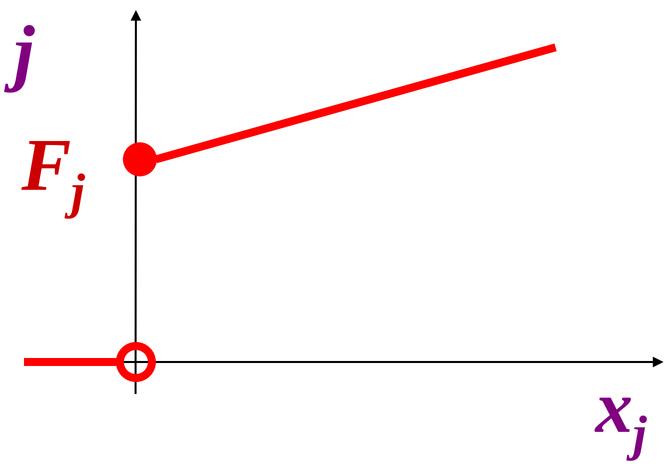
$$x_j \geq M_j t_j \; ; \quad x_j \leq G t_j \; ; \quad t_j \in \{0,1\}$$

with G very large positive number.

- If $t_j = 0$: $x_j \leq 0$ and $x_j \geq 0$, hence: $x_j = 0$;
- If $t_j = 1$: $x_j \leq G$ and $x_j \geq M_j$, hence: $x_j \geq M_j$

Fixed Production Cost

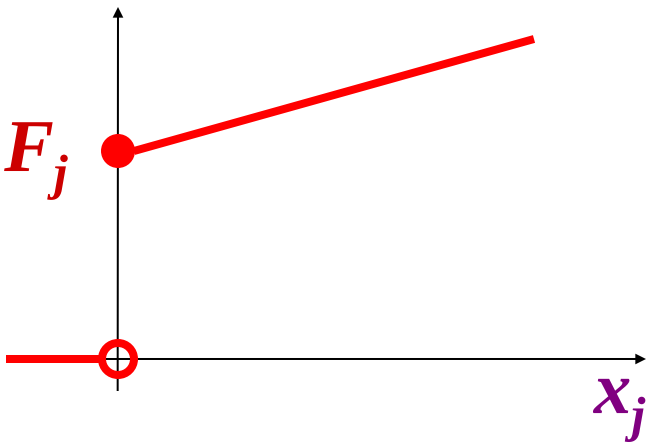
- **Production Problem concerning n items.**
- **The cost of item j ($j = 1,\ldots, n$) is given by:**
  - **0,** **if item j is not produced ($x_j = 0$)**
  - **$F_j + p_j x_j$** **if item j is produced ($x_j > 0$)**



- **F_j = fixed cost (machine)**
- **p_j = unit production cost**

- **discontinuity at the origin (not algebraic function)**

Fixed Production Cost (2)



$$y_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases}$$

$(j = 1, \ldots, n)$

Min $\Sigma_{j=1,n} (F_j\, y_j + p_j\, x_j)$

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \\ y &\in \{0,1\} \end{aligned}$$

Logical constraints (*if* …)

Variables *x* and *y* must be connected through linear constraints

Fixed Production Cost (3)

$$y_j = \begin{cases} 1 & \text{if} \quad x_j > 0 \\ 0 & \text{if} \quad x_j = 0 \end{cases} \quad \Rightarrow \quad x_j \le M y_j \quad (\,j = 1, \ldots, n)$$

con $M >> 1$ ($\cong +\infty$)

- **Satisfaction of the constraint:**
  - **if $x_j > 0$ y_j must be = 1 ($x_j \le$ M)**
  - **if $x_j = 0$ y_j can be = 0 or 1 ($0 \le 0$ or $0 \le M$)**

or

  - **if $y_j = 0$ x_j must be = 0 ($x_j \le 0$) (since $x_j \ge 0$)**
  - **if $y_j = 1$ x_j can be = 0 or > 0 ($x_j \le M$)**

- **The constraint imposes only a part of the logical relation**

Fixed Production Cost (4)

$$\text{Min } \sum_{j=1,n} (F_j y_j + p_j x_j)$$

$$Ax \geq b$$

$$x_j \leq My_j$$

$$x_j \geq 0, \quad y_j \in \{0, 1\} \qquad (j = 1, \ldots, n)$$

MILP Model (Mixed Integer Linear Programming)

- **It is not necessary to impose also the other part of the logical relation:**
- **a feasible solution with $x_j = 0$ and $y_j = 1$ cannot be optimal (an alternative feasible solution with a smaller cost exists: $x_j = 0$, $y_j = 0$)**

Discrete Variables

* Variable x must have a value among k (with $k > 1$) given different values:

$$x \in S = \{s_1, s_2, \ldots, s_k\}$$

* Let t_h ($h = 1, 2, \ldots, k$) be a *binary* variable such that:

$$t_h = 1 \text{ if } (x = s_h), \text{ and } t_h = 0 \text{ if } (x \neq s_h).$$

* $x = \Sigma_{h=1,k} \, s_h \, t_h$

with $\Sigma_{h=1,k} \, t_h = 1$

and $t_h \in \{0, 1\}$ ($h = 1, \ldots, k$)

Discrete Variables (2)

* **Example**

$x \in S = \{0.2, 0.4, \ldots, 2.0\}$ $(k = 10)$

- $t_h \in \{0, 1\}$ with $h = 1, 2, \ldots, 10$

- $x = 0.2\, t_1 + 0.4\, t_2 + 0.6\, t_3 + \ldots + 2.0\, t_{10}$

with

$$\Sigma_{h = 1,10}\ t_h = 1$$

* **Alternative technique:**

$x = 0.2\, y$ with $y \geq 1$, $y \leq 10$, y *integer*

Transformation of an ILP model into a Binary Linear Programming (BLP) model

* x integer variable with $x \geq 0$, $x \leq k$

* Introduce $(k + 1)$ binary variables t_h, with $h = 0, \ldots, k$
  ($t_h = 1$ if $x = h$, and $t_h = 0$ otherwise)

$$x = \Sigma_{h = 0, k}\, h\, t_h$$

$$\Sigma_{h = 0, k}\, t_h = 1 \quad \text{with } t_h \in \{0, 1\}\ h = 0, \ldots, k$$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- ***Example:*** x **integer variable with** $0 \leq x \leq 27$
- $z = log_2 \ (27 + 1)$, $q = \lceil z \rceil = 5$
- $x = t_1 + 2\, t_2 + 4\, t_3 + 8\, t_4 + 16\, t_5$
- **We must impose the constraint:**

$$t_1 + 2\, t_2 + 4\, t_3 + 8\, t_4 + 16\, t_5 \leq 27 \quad (**)$$

(for $t_1 = t_2 = t_3 = t_4 = t_5 = 1$, we have: $x = 31$)

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- ***Example, Alternative Transformation:***
- x **integer variable with** $0 \leq x \leq 27, \quad q = 5$

$12 = 16 - (31 - 27)$

- $x = t_1 + 2\,t_2 + 4\,t_3 + 8\,t_4 + 12\,t_5$
- **We have not to impose the constraint:**

$$t_1 + 2\,t_2 + 4\,t_3 + 8\,t_4 + 12\,t_5 \leq 27 \qquad (**)$$

(for $t_1 = t_2 = t_3 = t_4 = t_5 = 1$, we have: $x = 27$)

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- x integer variable with $x \geq b$, $x \leq k$ with $b \neq 0$

introduce q binary variables t_h, with $h = 1, \ldots, q$

$$x = \Sigma_{h = 1, q}\ 2^{h-1} t_h + b \quad , q = \lceil z \rceil, z = \log_2 (k - b + 1)$$

$$\Sigma_{h = 1, q}\ 2^{h-1} t_h \leq k - b \qquad t_h \in \{0, 1\}\ h = 1, \ldots, q \quad (**)$$

Example: $5 \leq x \leq 8, \quad q = 2, \quad x = 5 + t_1 + 2\, t_2$

Transformation of an ILP model into a Binary Linear Programming (BLP) model

- x integer variable with $x \geq 0$, $x \leq k$

Alternative technique (binary expression of an integer):

introduce q binary variables t_h, with $h = 1, \ldots, q$

$$x = \sum_{h=1,q} 2^{h-1} t_h = t_1 + 2\,t_2 + 4\,t_3 + \ldots + 2^{q-1} t_q$$

$$\sum_{h=1,q} 2^{h-1} t_h \leq k \qquad t_h \in \{0, 1\}\; h = 1, \ldots, q \quad (**)$$

$$q = \lceil z \rceil \qquad \text{with} \quad z = \log_2 (k+1)$$