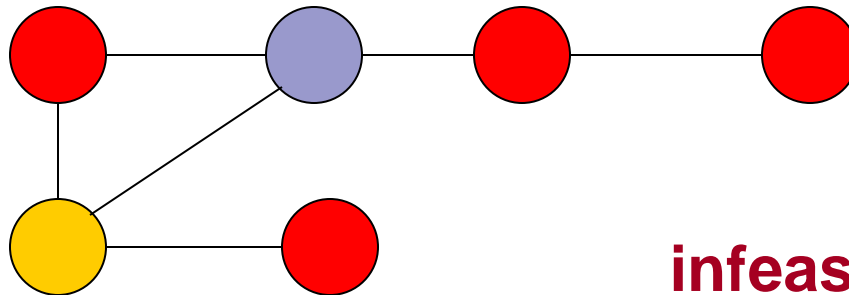


Vertex Coloring Problem (VCP): Models

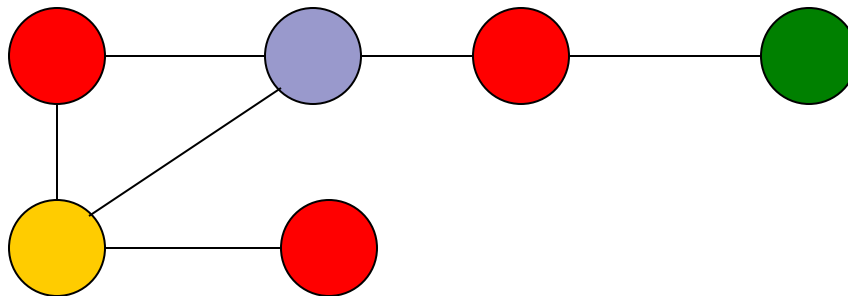
- Given an undirected graph $G = (V, E)$, with $n = |V|$ and $m = |E|$, assign a color to each vertex in such a way that colors on adjacent vertices are different, and the number of colors used is minimized.
- *chromatic number* $\chi(G)$: minimum number of colors which can be used.
- A feasible coloring which uses k colors is a *k-coloring*.



infeasible coloring

Vertex Coloring Problem (VCP)

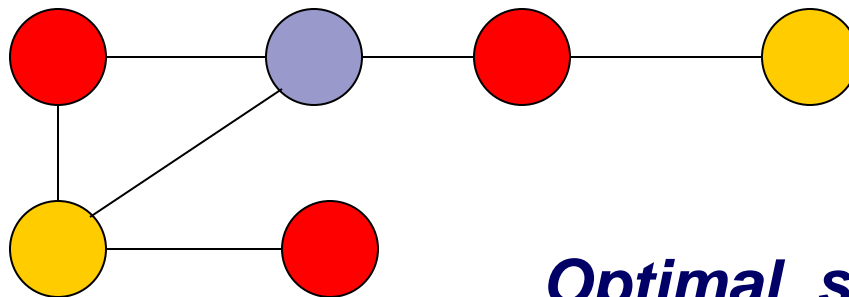
- Given an undirected graph $G = (V, E)$, with $n = |V|$ and $m = |E|$, assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized.
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4-coloring

Vertex Coloring Problem (VCP)

- Given an undirected graph $G = (V, E)$, with $n = |V|$ and $m = |E|$, assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized.
- *chromatic number* $\chi(G)$: minimum number of colors which can be used.
- A feasible coloring which uses k colors is a *k-coloring*.



3-coloring

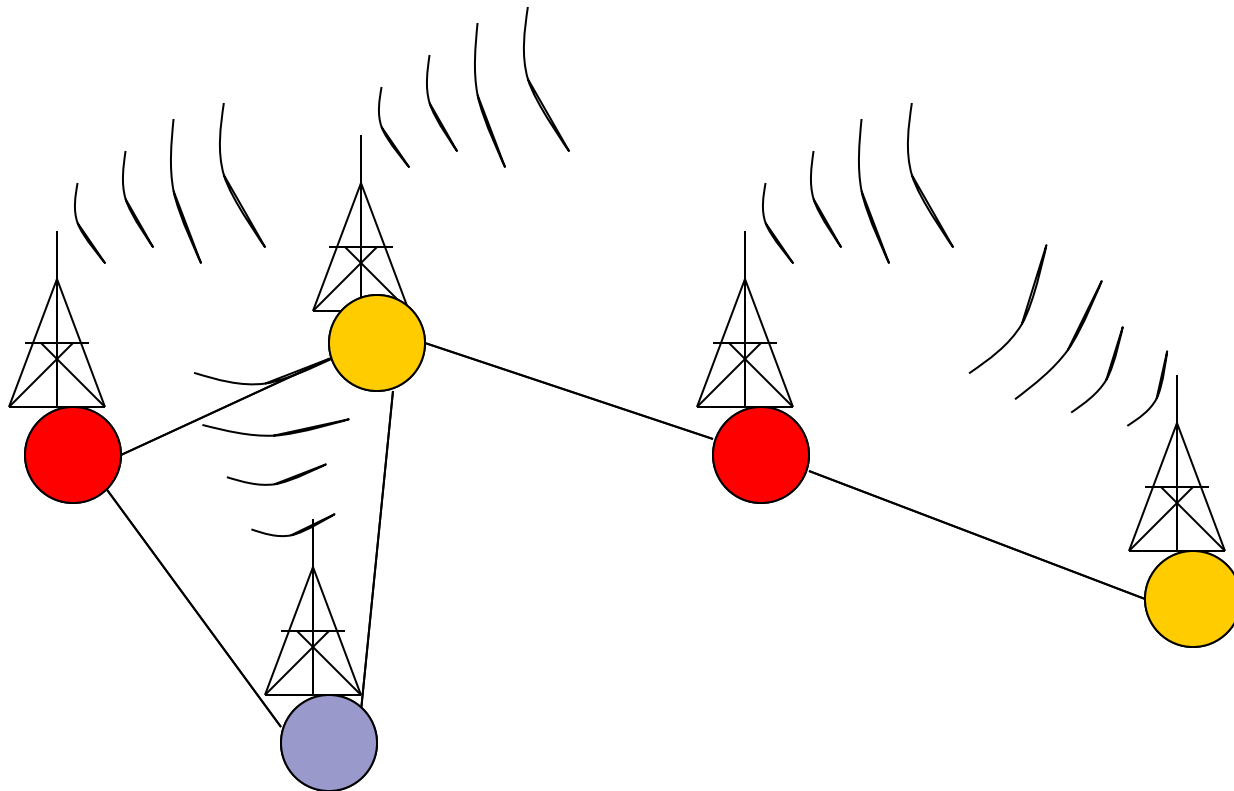
Optimal solution ($\chi(G)=3$)

Vertex Coloring Problem (VCP)

- **VCP is known to be NP-Hard** (Garey and Johnson, 1979).
- **If k is fixed ($k < n$) the feasibility problem is NP-Hard.**
- **Real-world applications:**
 - air traffic flow management;
 - register allocation;
 - **frequency assignment;**
 - communication networks;
 - crew scheduling;
 - train platforming;
 - printed circuit testing;
 - round-robin sports scheduling;
 - course timetabling;
 - geographical information systems;
 - ...

Application: Frequency Assignment

- Problem: given a set of broadcast emitting **stations** (vertices), assign a **frequency** (color) to each station so that adjacent (and possibly interfering) stations use different frequencies and the number of used frequencies is minimized.



Surveys

- Galinier, Hertz
(*Computers & Operations Research*, 2006);
- Chiarandini, Dumitrescu, Stutzle (*Handbook of Approximation Algorithms and Metaheuristics*, Gonzalez ed., Chapman & Hall/CRC, 2007);
- Johnson, Mehrotra, Trick
(*Discrete Applied Mathematics*, 2008);
- Malaguti, T.
(*International Trans. in O. R.*, 2010).

Web Page:

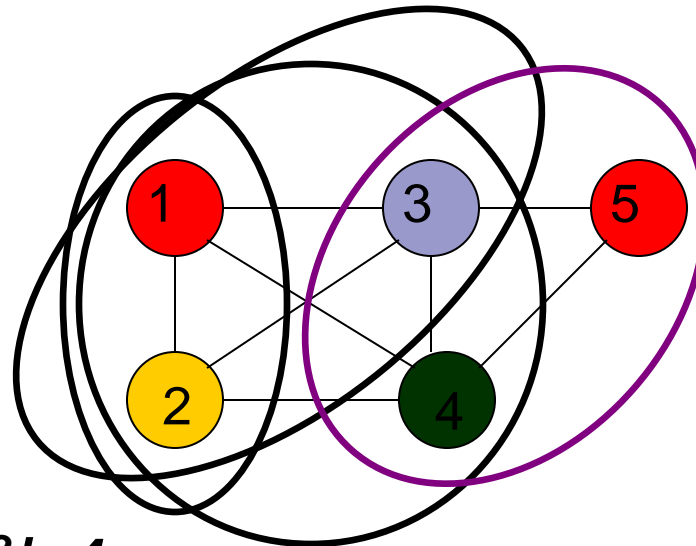
Bibliography on VCP (Chiarandini, Gualandi)

The Clique Lower Bound

- A **clique** K of a graph G is a complete subgraph of G .
- A clique is **maximal** if no vertex can be added still having a clique.
- The cardinality ω of the maximum (cardinality) clique is a **Lower Bound** for VCP. Computing ω is NP-Hard.

clique k , $|k|=2$

clique k^1 , $|k^1|=3$



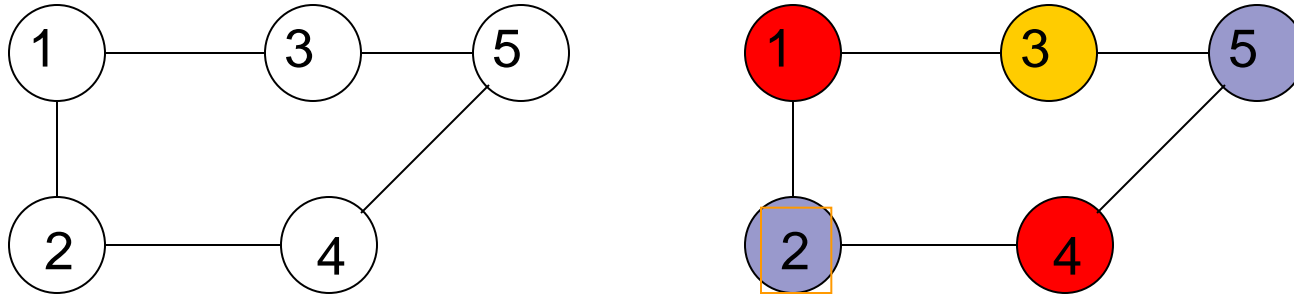
maximal clique k^2 , $|k^2|=4$

maximal clique k^3 , $|k^3|=3$

$$LB = \omega = 4$$

$$\chi(G) = 4$$

The Clique Lower Bound



cardinality of any clique (and of the maximum clique) $|K| = |K_{max}| = 2$:

$$LB = \omega = 2$$

chromatic number $\chi(G) = 3$

The worst case performance ratio $\omega / \chi(G)$ is arbitrarily bad

Maximal Clique

- The cardinality of any (maximal) clique of graph G represents a *Lower Bound* for the problem.
- A fast *greedy algorithm* (D. Johnson, *J. Comp. Syst. Sci.* 1974) can be used to compute a maximal clique K of $G(V,E)$:
Given an ordering of the vertices, consider the candidate vertex set W . Set $W = V$, $K = \emptyset$, and iteratively (while $W \neq \emptyset$):
 - * Choose the **vertex v** of W of maximum degree and add it to the current clique K .
 - * Remove from W **vertex v** and all the vertices not adjacent to the current clique K .
- Different orderings of the vertices generally produce different maximal cliques.

ILP models for VCP: Model VCP-ASSIGN (A)

- Binary variables: $x_{ih} = \begin{cases} 1 & \text{if vertex } i \text{ has color } h \\ 0 & \text{otherwise} \end{cases} \quad \begin{matrix} i = 1, \dots, n \\ h = 1, \dots, n \end{matrix}$
 $y_h = \begin{cases} 1 & \text{if color } h \text{ is used} \\ 0 & \text{otherwise} \end{cases} \quad h = 1, \dots, n$

$$\min \sum_{h=1}^n y_h \quad (1)$$

$$\sum_{h=1}^n x_{ih} = 1 \quad i = 1, \dots, n \quad (2)$$

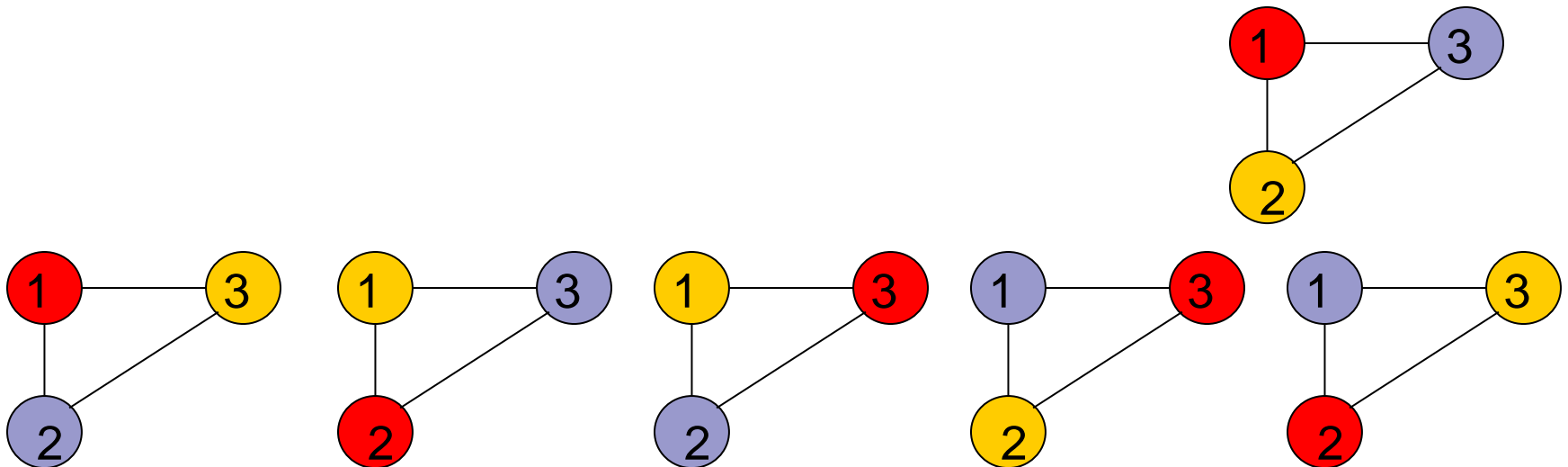
$$x_{ih} + x_{jh} \leq y_h \quad \forall i, j : (i, j) \in E \quad h = 1, \dots, n \quad (3)$$

$$x_{i,h} \in \{0,1\} \quad i = 1, \dots, n \quad h = 1, \dots, n$$

$$y_h \in \{0,1\} \quad h = 1, \dots, n \quad (4)$$

Model VCP-ASSIGN (A) is a “weak” model (2)

- “**Symmetry Property**”:
- Every solution of value k ($k < n$) has $\binom{n}{k} k!$ equivalent representations,
 $k!$ once the k colors have been chosen.
- **Example $k = 3$** ($k! = 6$)



A stronger ILP model (A') for VCP?

- Binary variables:

$$x_{ih} = \begin{cases} 1 & \text{if vertex } i \text{ has color } h & i=1,\dots,n \\ 0 & \text{otherwise} & h=1,\dots,n \end{cases}$$

$$y_h = \begin{cases} 1 & \text{if color } h \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

$$\min \sum_{h=1}^n y_h \quad (1)$$

$$\sum_{h=1}^n x_{ih} = 1 \quad i = 1,\dots,n \quad (2)$$

$$\sum_{i \in K} x_{ih} \leq y_h \quad \forall \max \text{ clique } K \subseteq V, \quad h = 1,\dots,n \quad (3)$$

$$x_{i,h} \in \{0,1\} \quad i = 1,\dots,n \quad h = 1,\dots,n$$

$$y_h \in \{0,1\} \quad h = 1,\dots,n \quad (4)$$

The number of constraints (3) grows exponentially with n .

Let K' be the *maximum clique* of G , and $|K'| = k$.

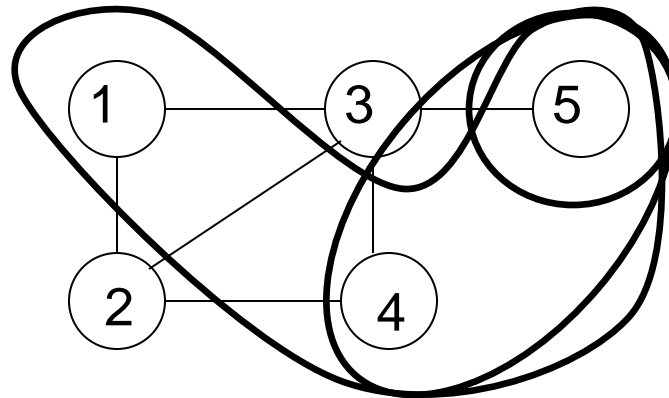
The continuous relaxation of (A') has the useless solution of value k :

$$y_1 = 1, \dots, y_k = 1; \quad y_h = 0 \quad h = k+1, \dots, n$$

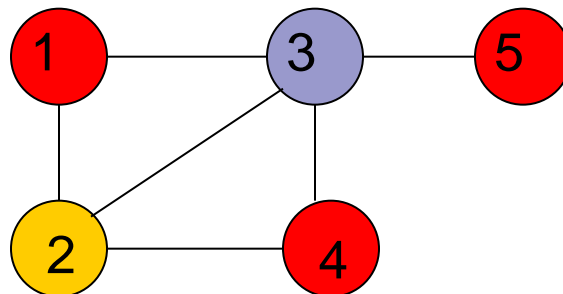
$$x_{i1}, \dots, x_{ik} = 1/k \quad i=1,\dots,n \quad x_{ih} = 0 \quad i=1,\dots,n \quad h=k+1,\dots,n$$

Independent Sets

- An *Independent Set* (or *Stable Set*) of $G = (V, E)$ is a subset of V such that there is no edge in E connecting a pair of vertices.
- It is *maximal* if no vertex can be added still having an independent set.



For VCP: all the vertices of an independent set can have the same color
Feasible coloring -> **partitioning** of the graph into independent sets.



Set Partitioning Formulation for VCP

(Mehrotra, Trick; INFORMS J. on. Comp. 1996)

- *Feasible coloring* -> *partition* of the graph into *independent sets*.
- **IS** = family of all the Independent Sets of graph **G**
- Binary variables: $x_I = \begin{cases} 1 & \text{if Independent Set } I \text{ is given a color} \\ 0 & \text{otherwise} \end{cases}$

s.t.
$$\min \sum_{I \in IS} x_I \quad (1)$$

$$\sum_{I: v \in I} x_I = 1 \quad \forall v \in V \quad (2)$$

$$x_I \in \{0,1\} \quad \forall I \in IS \quad (3)$$

Constraints (2) can be replaced by:
$$\sum_{I: v \in I} x_I \geq 1 \quad \forall v \in V \quad (2')$$

Set Covering Formulation SC -VCP

$$\begin{array}{ll} \min & \sum_{I \in IS} x_I \quad (1) \\ \text{s.t.} & \end{array}$$

$$\sum_{I: v \in I} x_I \geq 1 \quad \forall v \in V \quad (2')$$

$$x_I \in \{0,1\} \quad \forall I \in IS \quad (3)$$

- If a vertex is assigned more than one color, a feasible solution of the same value can be obtained by using any of these colors for the vertex.
- ***IS*** can be defined as the family of all the ***maximal Independent Sets*** of graph G .

Set Covering Formulation SC -VCP

$$\begin{array}{ll} \min & \sum_{I \in IS} x_I \quad (1) \\ \text{s.t.} & \end{array}$$

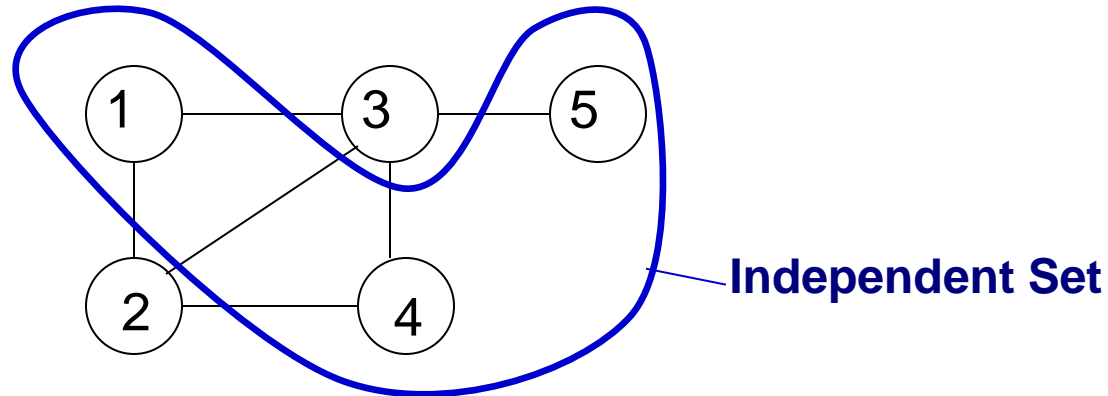
$$\sum_{I: v \in I} x_I \geq 1 \quad \forall v \in V \quad (2')$$

$$x_I \in \{0,1\} \quad \forall I \in IS \quad (3)$$

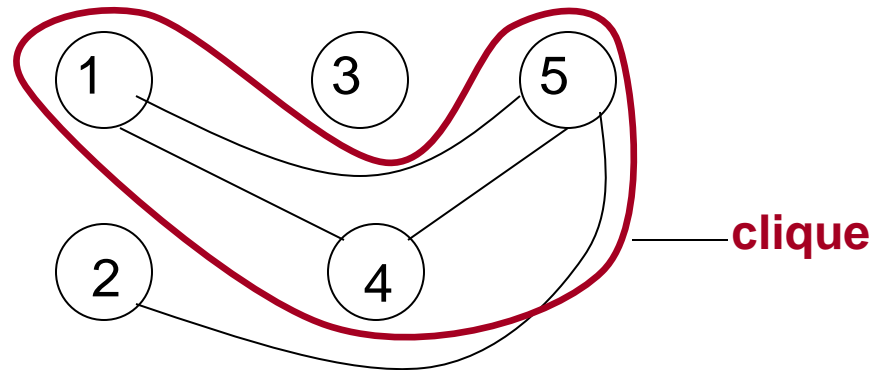
- The *LP Relaxation* of this formulation leads to *tight lower bounds*, and *symmetry* in the solution is *avoided*, but the number of maximal independent sets (i.e. the number of “variables”, or “columns”) can be *exponential* w.r.t. the number of vertices $n \rightarrow$
- The corresponding SCP is difficult to solve to optimality.

Independent Sets and Cliques

- Given a graph $G = (V, E)$



Define its “complement” $\bar{G} = (V, \bar{E})$, where $\bar{E} = \{(i, j): (i, j) \notin E\}$



independent set of $G \rightarrow$ clique of \bar{G} (and viceversa)

clique of $G \rightarrow$ independent set of \bar{G} (and viceversa)

Additional ILP Formulations

- Williams and Yan (*INFORMS J. on Comp.*, 2001): VCP-ASSIGN plus “precedence constraints”.
- Lee (*J. of Comb. Opt.*, 2002), and Lee and Margot (*INFORMS J. on Comp.*, 2007): binary encoding formulation.
- Barbosa, Assis, do Nascimento (*J. of Comb. Opt.*, 2004): encodings based on acyclic orientations.
- Burke, Marecek, Parkes, Rudova (*Ann. of Oper. Res.*, 2010): “supernodal” formulation (transformation of the original VCP into a *Multicoloring Vertex Problem* having a smaller number of vertices and edges).