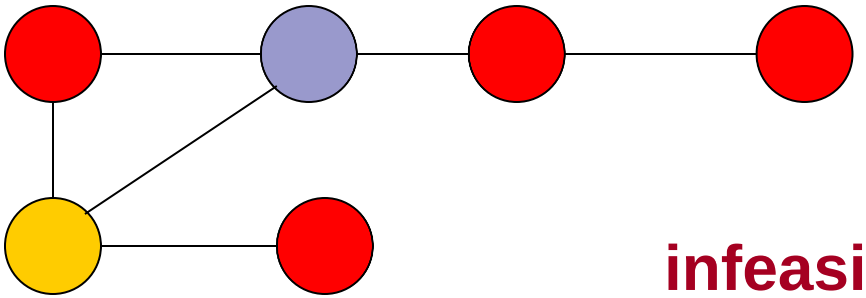
Vertex Coloring Problem (VCP): Models

- **Given an undirected graph $G = (V,E)$, with $n = |V|$ and $m = |E|$, assign a color to each vertex in such a way that colors on adjacent vertices are different, and the number of colors used is minimized.**
- ***chromatic number $\chi(G)$:* minimum number of colors which can be used.**
- **A feasible coloring which uses *k* colors is a *k-coloring*.**



infeasible coloring

Vertex Coloring Problem (VCP)

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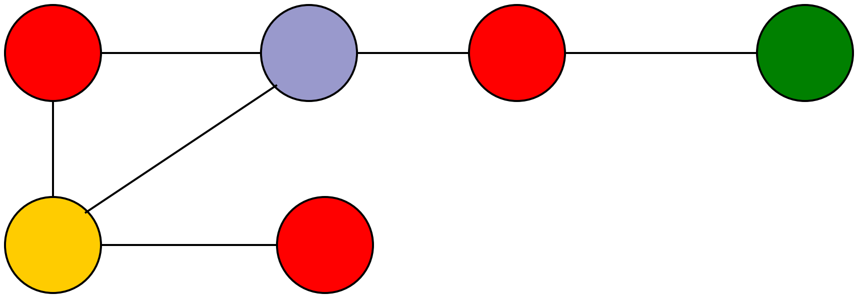
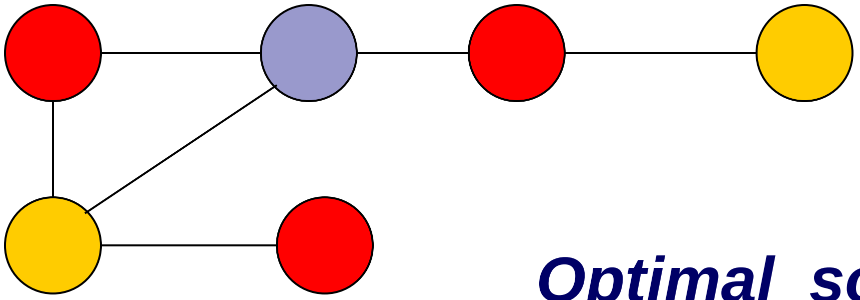


4-coloring

Vertex Coloring Problem (VCP)

- Given an undirected graph *G = (V,E)*, with *n = |V|* and *m = |E|*, assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized.
- *chromatic number χ(G):* minimum number of colors which can be used.
- A feasible coloring which uses *k* colors is a *k-coloring*.



3-coloring

Optimal solution (χ(G)=3)

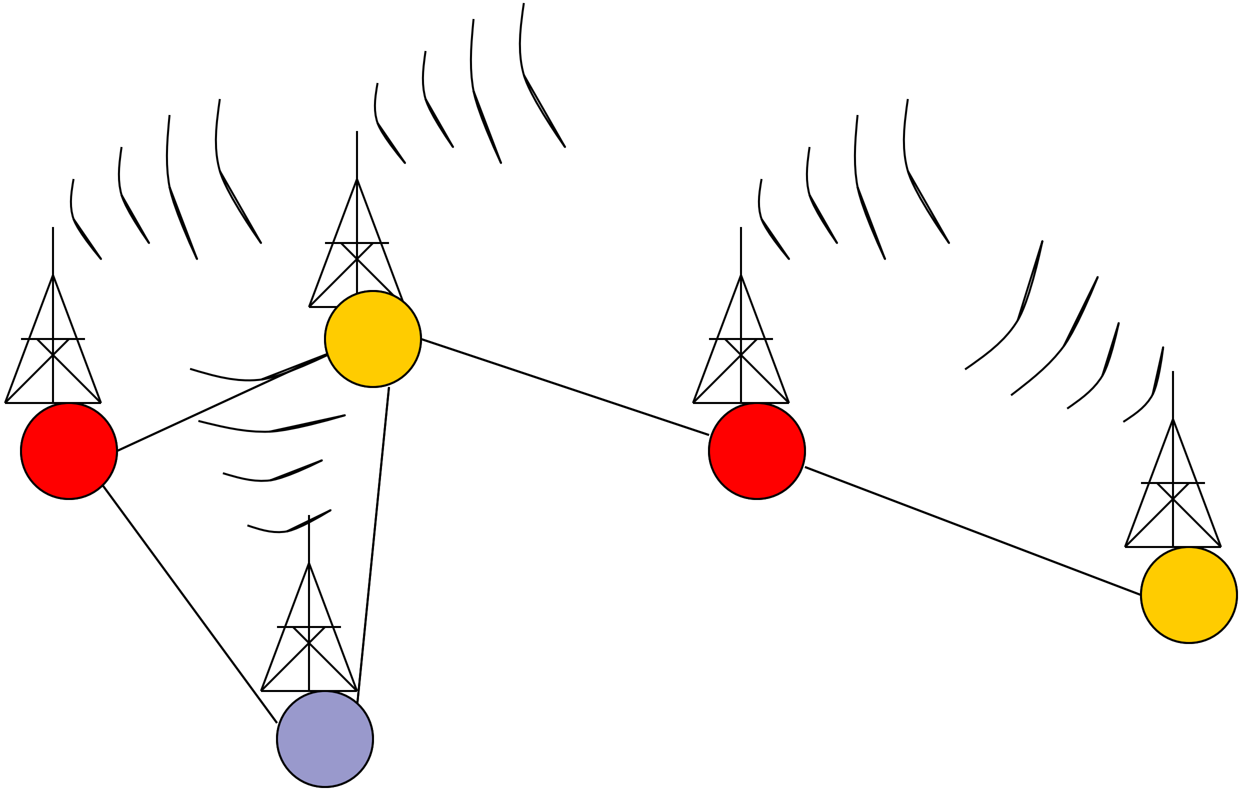
Vertex Coloring Problem (VCP)

- **VCP is known to be NP-Hard (Garey and Johnson, 1979).**
- **If k is fixed (k < *n*) the feasibility problem is NP-Hard.**

- **Real-world applications:**
  - air traffic flow management;
  - register allocation;
  - frequency assignment;
  - communication networks;
  - crew scheduling;
  - train platforming;
  - printed circuit testing;
  - round-robin sports scheduling;
  - course timetabling;
  - geographical information systems;
  - …

Application: Frequency Assignment

- **Problem: given a set of broadcast emitting stations (vertices), assign a frequency (color) to each station so that adjacent (and possibly interfering) stations use different frequencies and the number of used frequencies is minimized.**



Surveys

- **Galinier, Hertz**
  (*Computers & Operations Research*, 2006);
- **Chiarandini, Dumitrascu, Stutzle (*Handbook of Approximation Algorithms and Metaheuristics*, Gonzalez ed., Chapman & Hall/CRC, 2007);**
- **Johnson, Mehrotra, Trick**
  (*Discrete Applied Mathematics*, 2008);
- **Malaguti, T.**
  (*International Trans. in O. R.*, 2010).

Web Page:

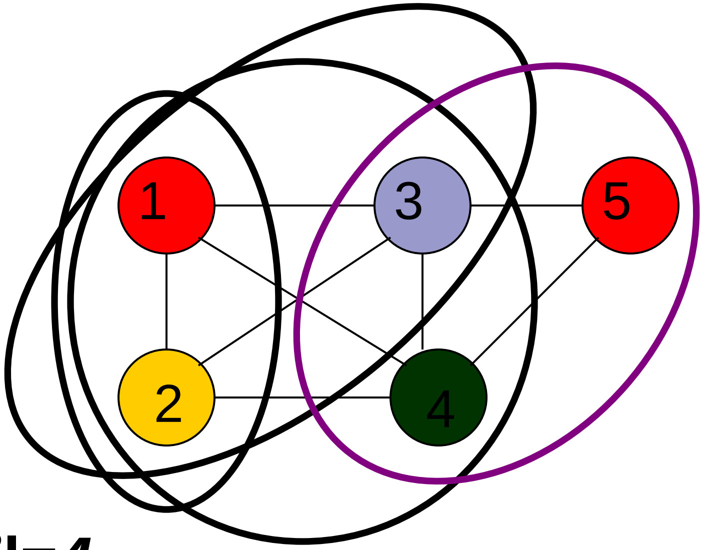
Bibliography on VCP (Chiarandini, Gualandi)

The Clique Lower Bound

- A *clique K* of a graph G is a complete subgraph of G.
- A clique is *maximal* if no vertex can be added still having a clique.
- The cardinality *ω* of the maximum (cardinality) clique is a *Lower Bound* for VCP. Computing *ω* is NP-Hard.

clique k, $|k|=2$

clique k^1, $|k^1|=3$



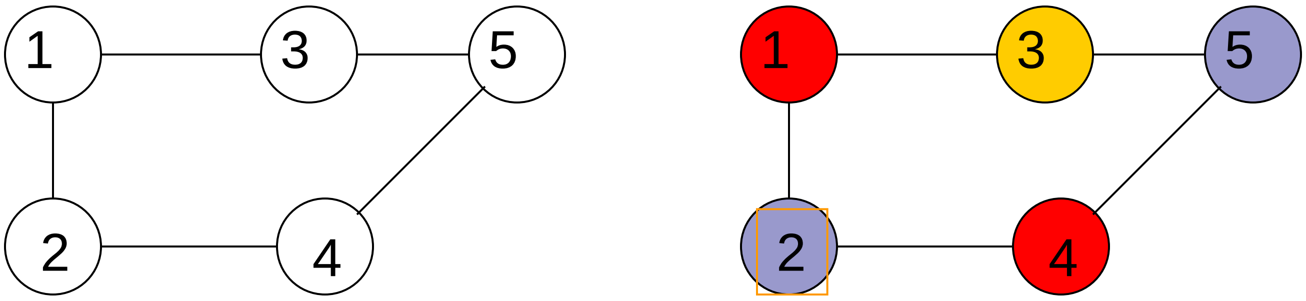
maximal clique k^2, $|k^2|=4$

maximal clique k^3, $|k^3|=3$

$LB = \omega = 4$

$\chi(G) = 4$

The Clique Lower Bound



cardinality of any clique (and of the maximum clique) $|K| = |K_{max}| = 2$:

$LB = \omega = 2$

chromatic number $\chi(G) = 3$

The worst case performance ratio $\omega / \chi(G)$ is arbitrarily bad

Maximal Clique

- **The cardinality of any (maximal) clique of graph *G* represents a *Lower Bound* for the problem.**

- **A fast *greedy algorithm* (D. Johnson, *J. Comp. Syst. Sci. 1974*) can be used to compute a maximal clique *K* of *G(V,E)*:**

  Given an ordering of the vertices, consider the candidate vertex set *W*. Set *W* = *V*, *K* = ∅, and iteratively

  (while W ≠ ∅):

  *** Choose the vertex *v* of *W* of maximum degree and add it to the current clique *K*.**

  *** Remove from *W* *vertex* *v* and all the vertices not adjacent to the current clique *K*.**

- **Different orderings of the vertices generally produce different maximal cliques.**

ILP models for VCP: Model VCP-ASSIGN (A)

- **Binary variables:**

$$x_{ih} = \begin{cases} \textbf{1 if vertex } \boldsymbol{i} \textbf{ has color } \boldsymbol{h} & \boldsymbol{i = 1,\ldots,n} \\ \textbf{0 otherwise} & \boldsymbol{h = 1,\ldots,n} \end{cases}$$

$$y_h = \begin{cases} \textbf{1 if color } \boldsymbol{h} \textbf{ is used} & \\ \textbf{0 otherwise} & \boldsymbol{h = 1,\ldots,n} \end{cases}$$

$$\min \sum_{h=1}^{n} y_h \qquad (1)$$

$$\sum_{h=1}^{n} x_{ih} = 1 \qquad i = 1,..,n \qquad (2)$$

$$x_{ih} + x_{jh} \le y_h \qquad \forall i, j : (i, j) \in E \quad h = 1,\ldots,n \qquad (3)$$

$$x_{i,h} \in \{0,1\} \qquad i = 1,\ldots,n \qquad h = 1,\ldots,n$$

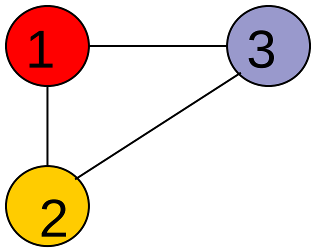
$$y_h \in \{0,1\} \qquad h = 1,\ldots,n \qquad (4)$$

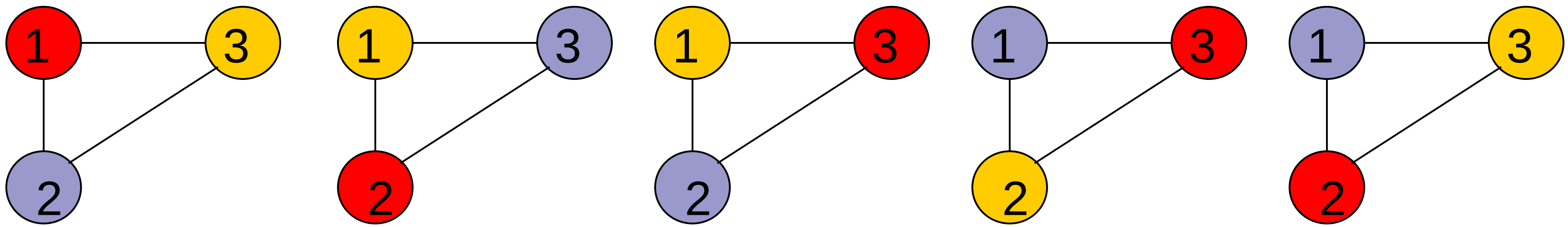
Model VCP-ASSIGN (A) is a “weak” model (2)

- “***Symmetry Property***”:
- Every solution of value k (k < *n*) has $\binom{n}{k}k!$ equivalent representations,

  k! once the *k* colors have been chosen.

- **Example k = 3** *(k! = 6)*





A stronger ILP model (A') for VCP?

- Binary variables:

$$x_{ih} = \begin{cases} 1 \text{ if vertex } i \text{ has color } h & i=1,\dots,n \\ 0 \text{ otherwise} & h=1,\dots,n \end{cases}$$

$$y_h = \begin{cases} 1 \text{ if color } h \text{ is used} \\ 0 \text{ otherwise} \end{cases}$$

$$\min \sum_{h=1}^{n} y_h \qquad (1)$$

$$\sum_{h=1}^{n} x_{ih} = 1 \qquad i = 1,..,n \qquad (2)$$

$$\sum_{i \in K} x_{ih} \le y_h \qquad \forall \max clique K \subseteq V, \quad h = 1,\dots,n \qquad (3)$$

$$x_{i,h} \in \{0,1\} \qquad i = 1,\dots,n \qquad h = 1,\dots,n$$

$$y_h \in \{0,1\} \qquad h = 1,\dots,n \qquad (4)$$

The number of constraints (3) grows exponentially with n.

Let *K'* be the *maximum clique* of G, and |*K'*| = *k*.

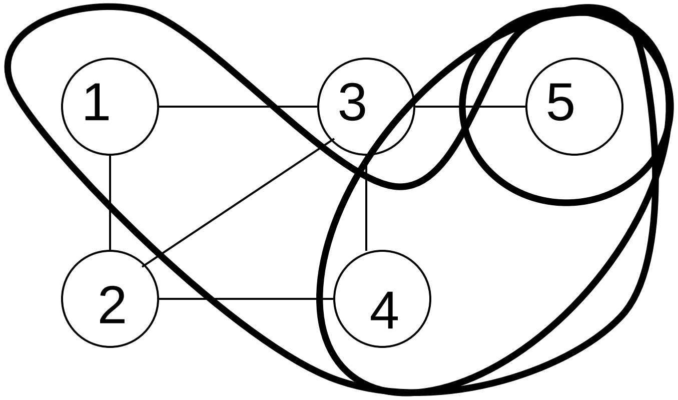
The continuos relaxation of (A') has the useless solution of value *k*:

$y_1 = 1, \dots, y_k = 1$; $y_h = 0$ $h = k+1, \dots, n$

$x_{i1}, \dots, x_{ik} = 1/k$ $i = 1, \dots, n$ $x_{ih} = 0$ $i = 1, \dots, n$ $h = k+1, \dots, n$

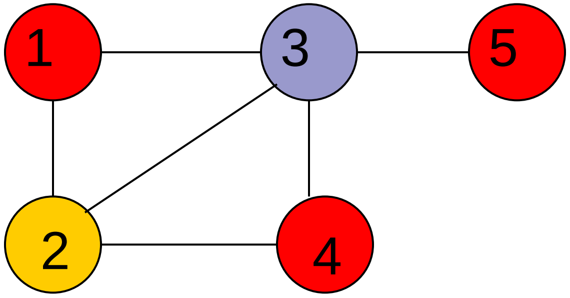
Independent Sets

- An *Independent Set* (or *Stable Set*) of $G = (V, E)$ is a subset of V such that there is no edge in E connecting a pair of vertices.
- It is *maximal* if no vertex can be added still having an independent set.



For VCP: all the vertices of an independent set can have the same color

Feasible coloring -> partitioning of the graph into independent sets.



Set Partitioning Formulation for VCP

(Mehrotra, Trick; INFORMS J. on. Comp. 1996)

- ***Feasible coloring* -> *partition* of the graph into *independent sets*.**
- ***IS*** = **family of all the Independent Sets of graph *G***
- **Binary variables**: $x_I = \begin{cases} 1 & \text{if Independent Set } I \text{ is given a color} \\ 0 & \text{otherwise} \end{cases}$

$$\min \sum_{I \in IS} x_I \qquad (1)$$

s.t.

$$\sum_{I : v \in I} x_I = 1 \qquad \forall v \in V \qquad (2)$$

$$x_I \in \{0,1\} \qquad \forall I \in IS \qquad (3)$$

Constraints (2) can be replaced by: $$\sum_{I : v \in I} x_I \geq 1 \qquad \forall v \in V \qquad (2')$$

Set Covering Formulation SC -VCP

$$\min \sum_{I \in IS} x_I \qquad (1)$$

s.t.

$$\sum_{I:v \in I} x_I \geq 1 \qquad \forall v \in V \qquad (2')$$

$$x_I \in \{0,1\} \qquad \forall I \in IS \qquad (3)$$

- **If a vertex is assigned more than one color, a feasible solution of the same value can be obtained by using any of these colors for the vertex.**
- ***IS*** **can be defined as the family of all the *maximal Independent Sets* of graph *G*.**

Set Covering Formulation SC -VCP

$$\min \sum_{I \in IS} x_I \qquad (1)$$

s.t.

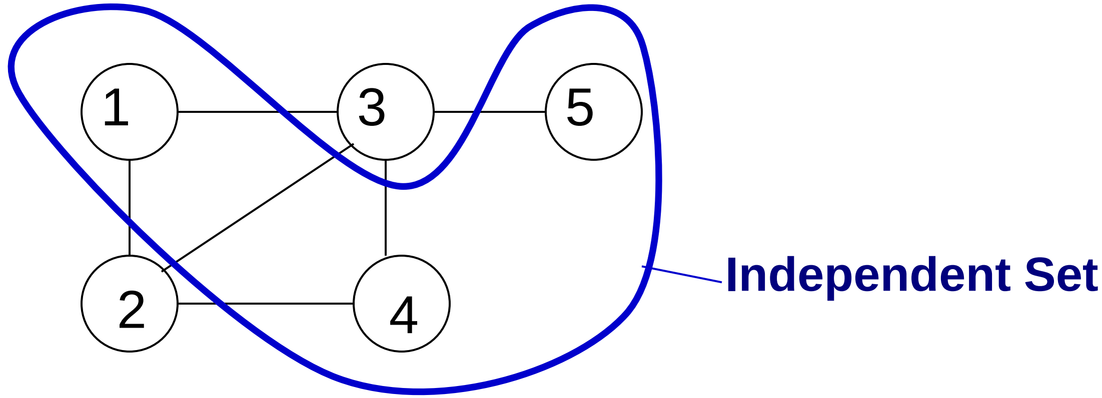
$$\sum_{I:v \in I} x_I \geq 1 \qquad \forall v \in V \qquad (2')$$

$$x_I \in \{0,1\} \qquad \forall I \in IS \qquad (3)$$

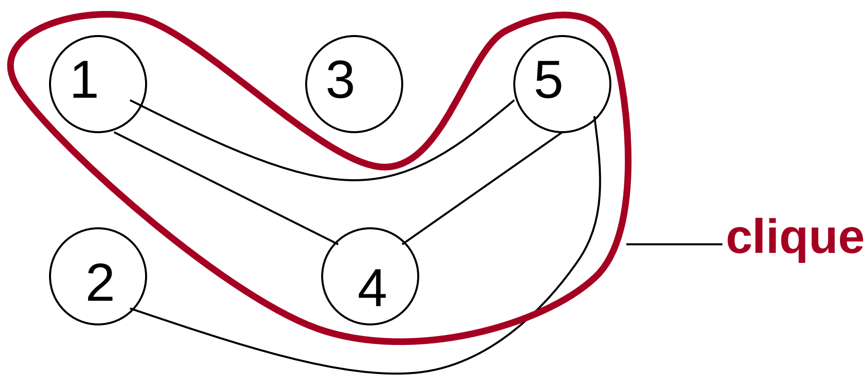
- **The *LP Relaxation* of this formulation leads to *tight lower bounds,* and *symmetry* in the solution is *avoided*, but the number of maximal independent sets (i.e. the number of "variables", or"columns") can be *exponential* w.r.t. the number of vertices *n* ->**
- **The corresponding SCP is difficult to solve to optimality.**

Independent Sets and Cliques

- **Given a graph $G = (V, E)$**



Define its "complement" $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{(i, j): (i, j) \notin E\}$



independent set of G → clique of $\overline{G}$ (and viceversa)

clique of G → independent set of $\overline{G}$ (and viceversa)

Additional ILP Formulations

- **Williams and Yan (INFORMS J. on Comp., 2001): VCP-ASSIGN plus "precedence constraints".**

- **Lee *(J. of Comb. Opt., 2002),* and Lee and Margot (*INFORMS J. on Comp., 2007*): binary encoding formulation.**

- **Barbosa, Assis, do Nascimiento (*J. of Comb. Opt., 2004*): encodings based on acyclic orientations.**

- **Burke, Marecek, Parkes, Rudova (*Ann. of Oper. Res., 2010*): "supernodal" formulation (transformation of the original VCP into a *Multicoloring Vertex Problem* having a smaller number of vertices and edges).**