Asymmetric Traveling Salesman Problem (ATSP): Models

- Given a DIRECTED GRAPH $G = (V,A)$ with
  - $V = \{1, \ldots, n\}$ vertex set
  - $A = \{(i, j) : i \in V, j \in V\}$ arc set (complete digraph)
  - c_{ij} = cost associated with arc $(i, j) \in A$ ($c_{ii} = \infty$, $i \in V$)
- Find a HAMILTONIAN CIRCUIT (Tour) whose global cost is minimum (Asymmetric Travelling Salesman Problem: ATSP).

Hamiltonian Circuit: circuit passing through each vertex of V exactly once.

A Hamiltonian circuit has $n = |V|$ arcs.

- ATSP is $\mathcal{NP}$-Hard in the strong sense.

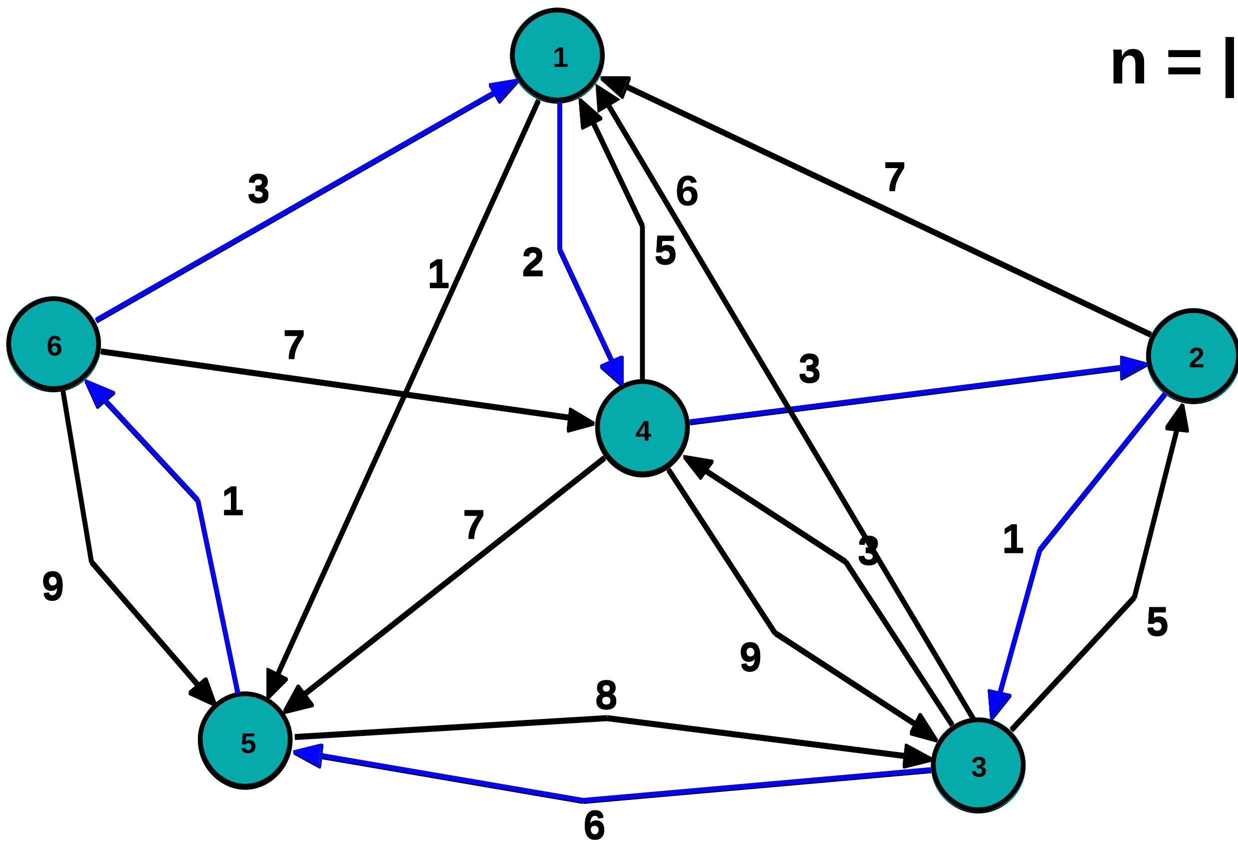
- If G is an undirected graph: **Symmetric TSP (STSP)**
  (special case of ATSP arising when $c_{ij} = c_{ji}$ for each $(i, j) \in A$)

- Any ATSP instance with n vertices can be transformed into an equivalent STSP instance with $2n$ nodes (Jonker-Volgenant, 1983; Junger-Reinelt-Rinaldi, 1995; Kumar-Li, 2007).

- If $G = (V, A)$ is a sparse graph: $c_{ij} = \infty$ for each $(i, j) \notin A$.

- **Feasible solutions?**

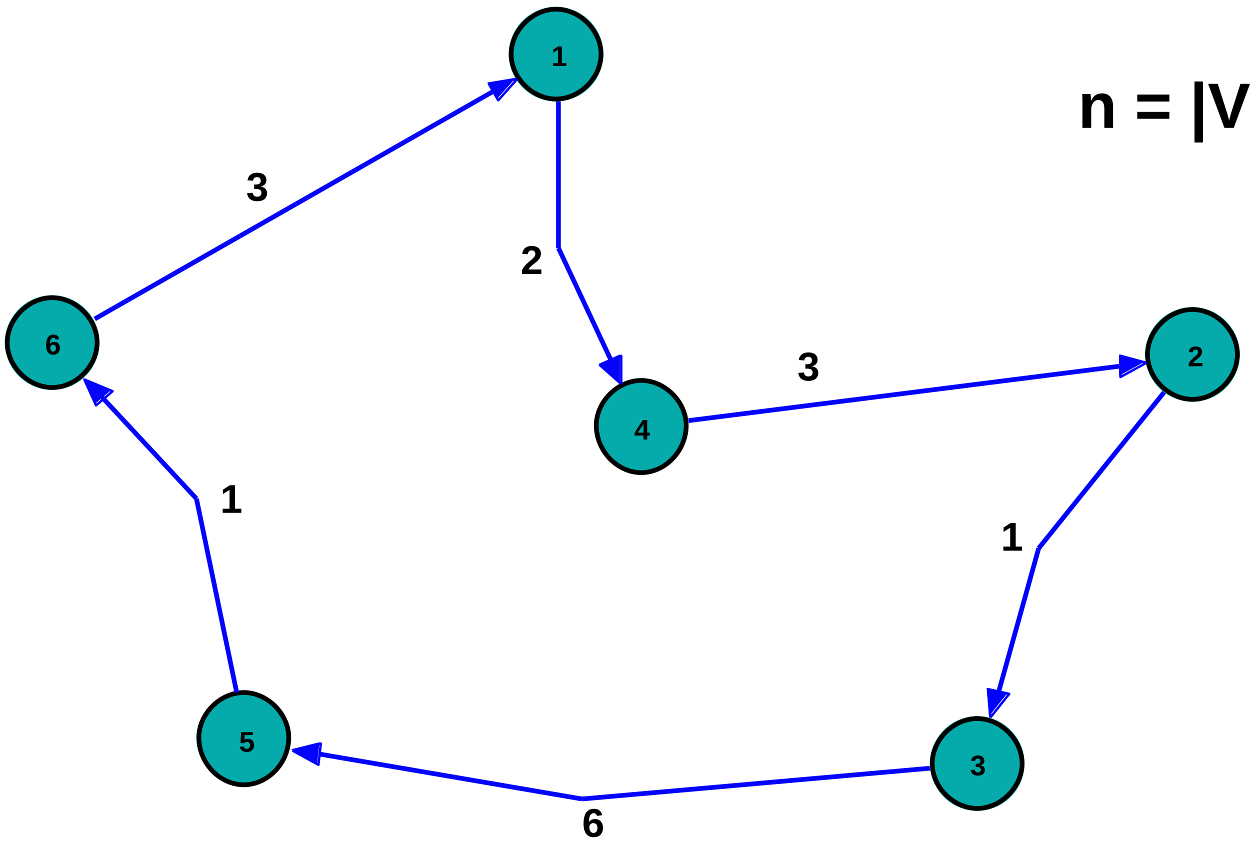
Example



n = |V| = 6

Optimal solution

Example



n = |V| = 6

Optimal solution

Optimal solution Cost = 2 + 3 + 1 + 6 +1 + 3 = 16

APPLICATIONS

* **Vehicle Routing** (sequencing the customers in each route in an urban area calls for the optimal solution of the ATSP corresponding to the depot and the customers in the route).

* **Scheduling** (optimal sequencing of jobs on a machine when the set-up costs depend on the sequence in which the jobs are processed).

- **Picking in an Inventory System** (sequence of movements of a crane to pick-up a set of items stored on shelves).
- …

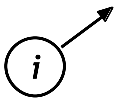
INTEGER LINEAR PROGRAMMING (ILP) FORMULATION

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the optimal tour} \\ 0 & \text{otherwise} \end{cases} \qquad i \in V, \; j \in V$$

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij} \, x_{ij}$$

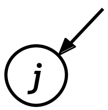
s.t.

out-degree constraints



$$\sum_{j \in V} x_{ij} = 1 \qquad i \in V$$

in-degree constraints

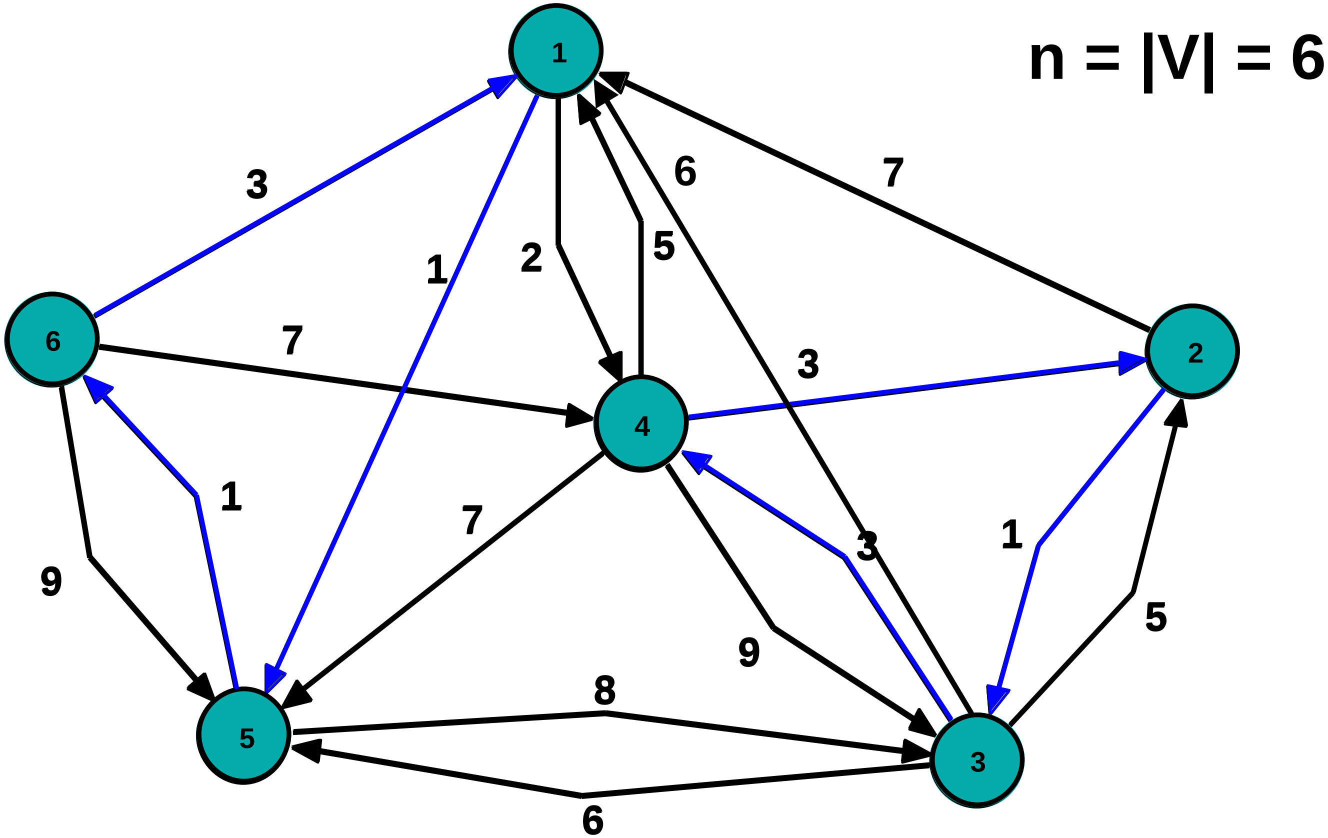


$$\sum_{i \in V} x_{ij} = 1 \qquad j \in V$$

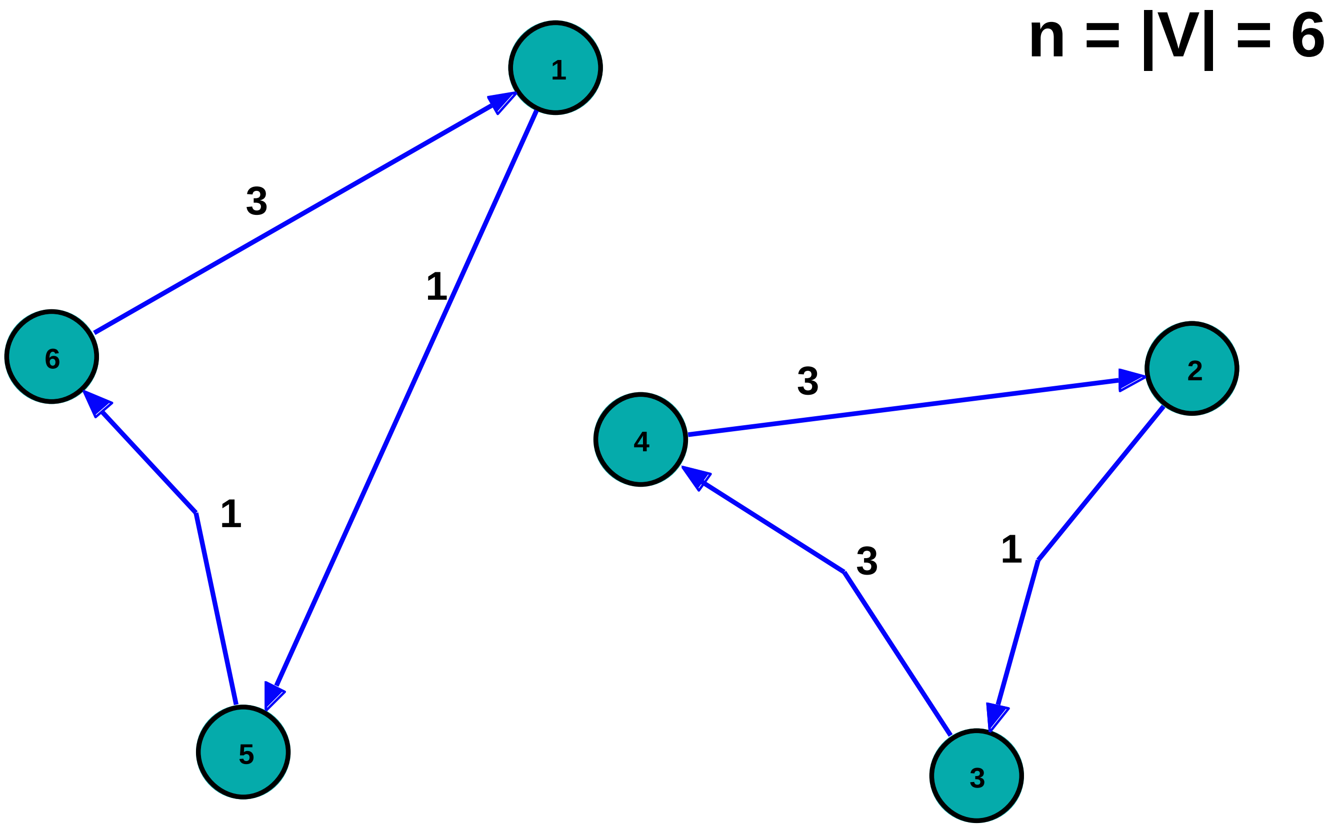
$$x_{ij} \in \{0, 1\} \qquad i \in V, j \in V$$

?

Example: only degree constraints imposed



Example: only degree constraints imposed



solution Cost = (1 + 1 + 3) + (3 + 1 + 3) = 12

Infeasible solution: two partial tours (subtours)

INTEGER LINEAR PROGRAMMING (ILP) FORMULATION

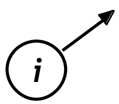
(Dantzig, Fulkerson, Johnson, Oper. Res. 1954)

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the optimal tour} \\ 0 & \text{otherwise} \end{cases} \qquad i \in V,\ j \in V$$

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij}\, x_{ij}$$

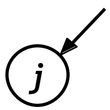
s.t.

out-degree constraints



$$\sum_{j \in V} x_{ij} = 1 \qquad i \in V$$

in-degree constraints



$$\sum_{i \in V} x_{ij} = 1 \qquad j \in V$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1 \qquad S \subset V,\ |S| \geq 2$$

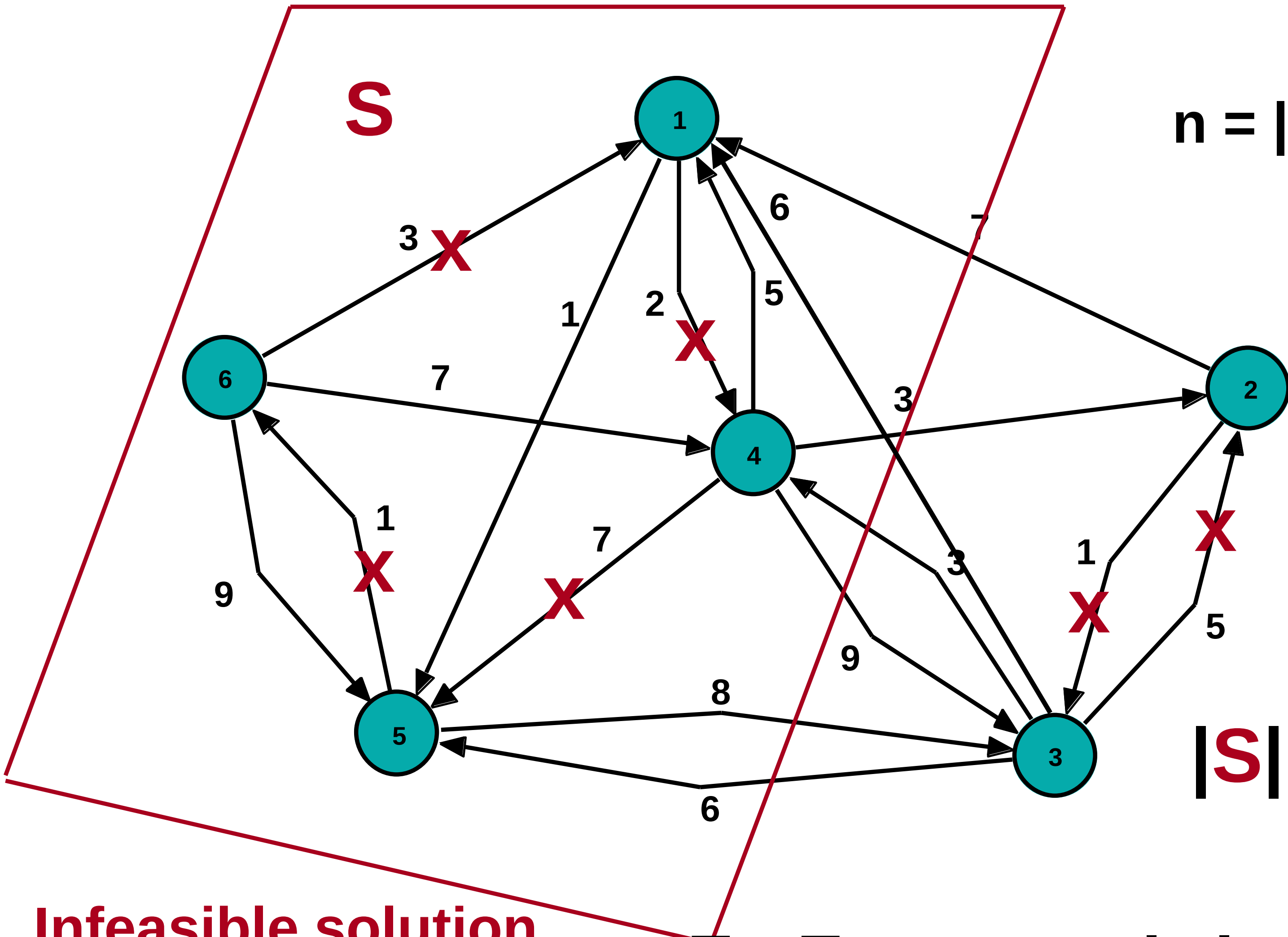
!!!

SUBTOUR ELIMINATION CONSTRAINTS

(forbid the partial tours; $O(2^n)$)

$$x_{ij} \in \{0, 1\} \qquad i \in V,\ j \in V$$

Example



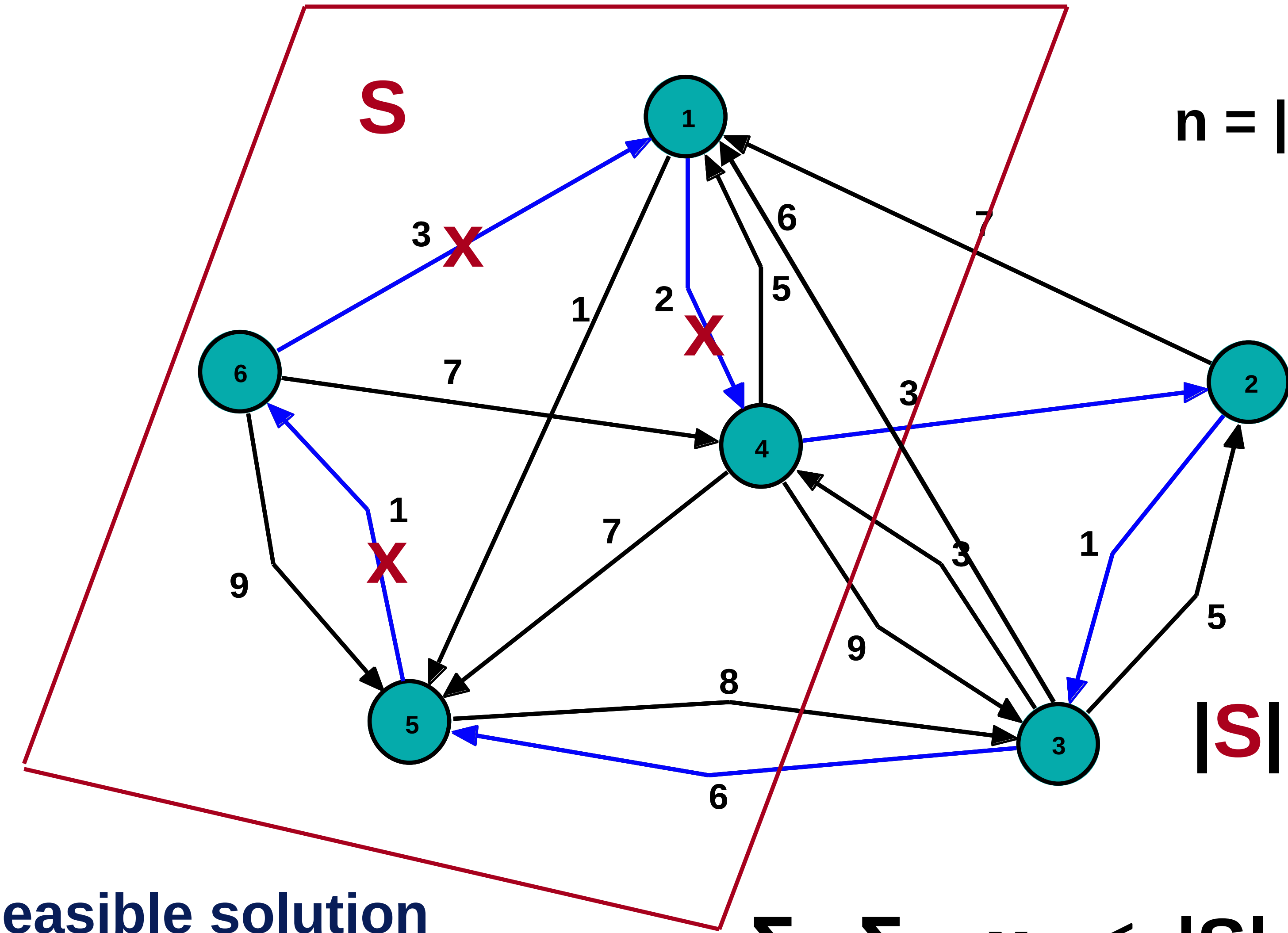
n = |V| = 6

|S| = 4

X Infeasible solution

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$$ NO!

Example



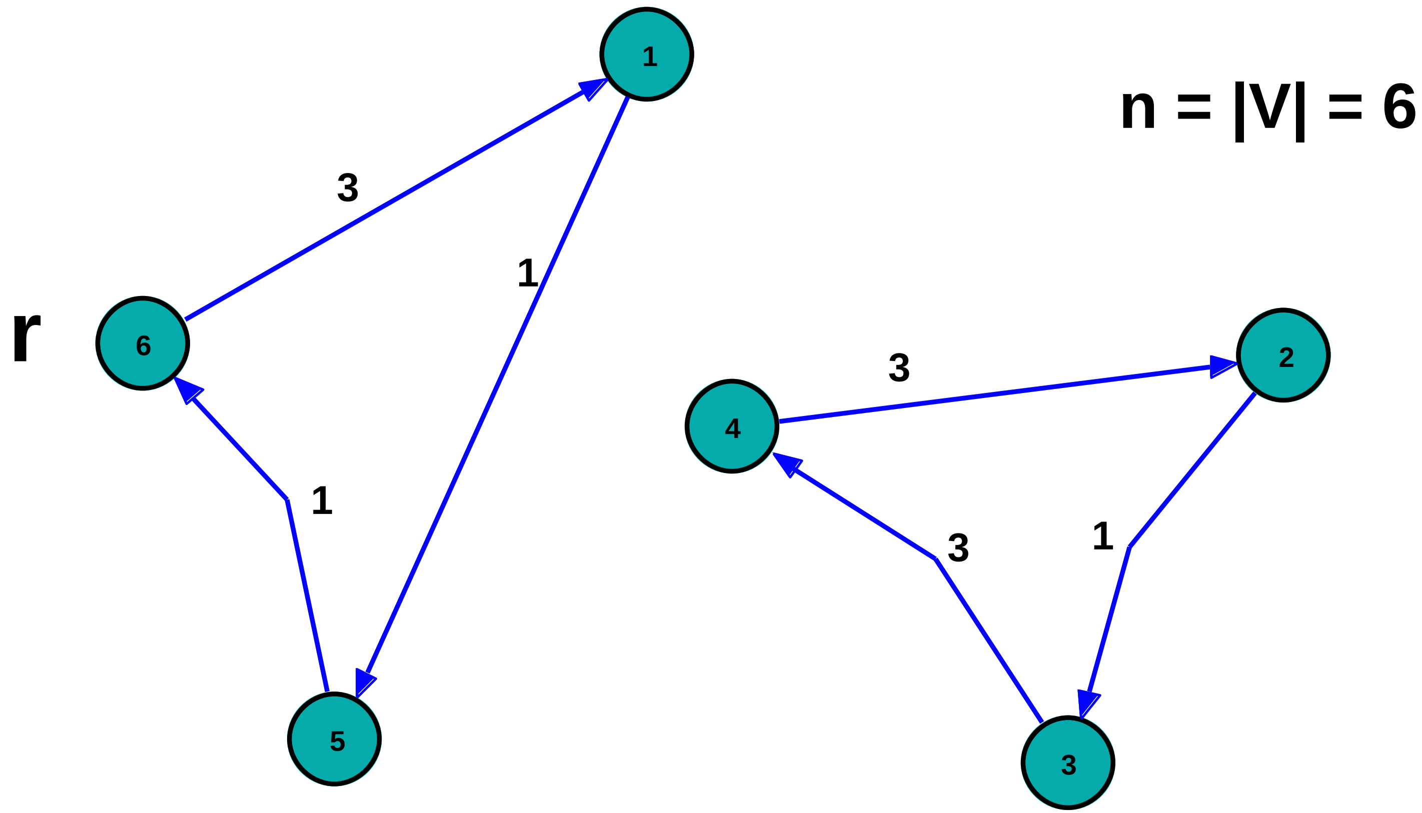
n = |V| = 6

|S| = 4

Feasible solution

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$$ **YES!**

Example: only degree constraints imposed



Infeasible solution: the vertices are not connected:

In a Hamiltonian circuit:

from any vertex (say r) we must reach all the other vertices (connectivity from r), and viceversa (connectivity toward r)

INTEGER LINEAR PROGRAMMING FORMULATION

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the optimal tour} \\ 0 & \text{otherwise} \end{cases} \qquad i \in V,\ j \in V$$

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij}$$

s.t.

$$\sum_{j \in V} x_{ij} = 1 \qquad i \in V$$

$$\sum_{i \in V} x_{ij} = 1 \qquad j \in V$$

$$\sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \geq 1 \qquad S \subset V,\ r \in S$$

(for a fixed $r \in V$)

$$x_{ij} \in \{0, 1\} \qquad i \in V, j \in V$$

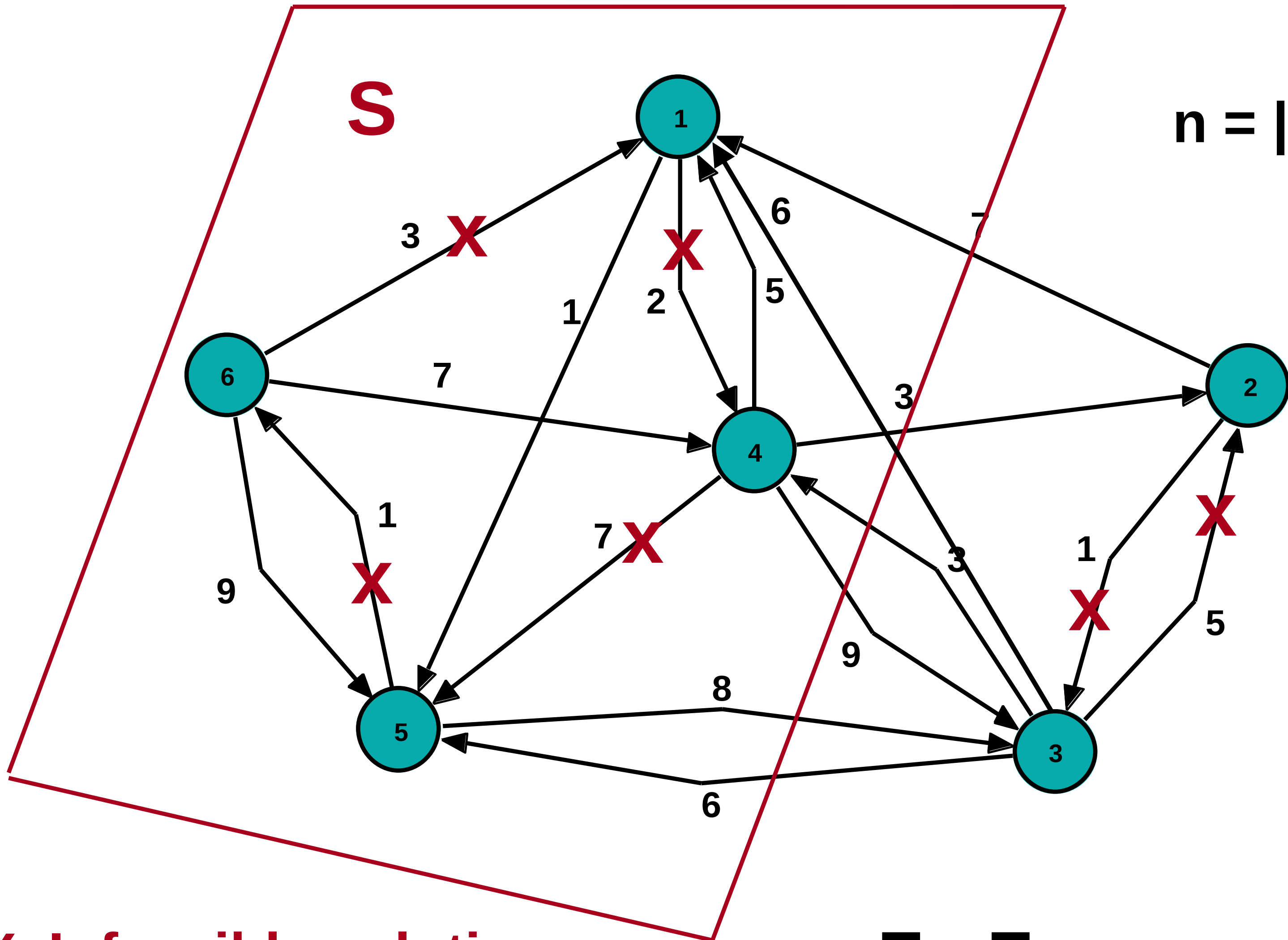
CONNECTIVITY CONSTRAINTS

(impose the connectivity of the solution; $O(2^n)$)

Cut inequalities

The two formulations are "equivalent"

Example

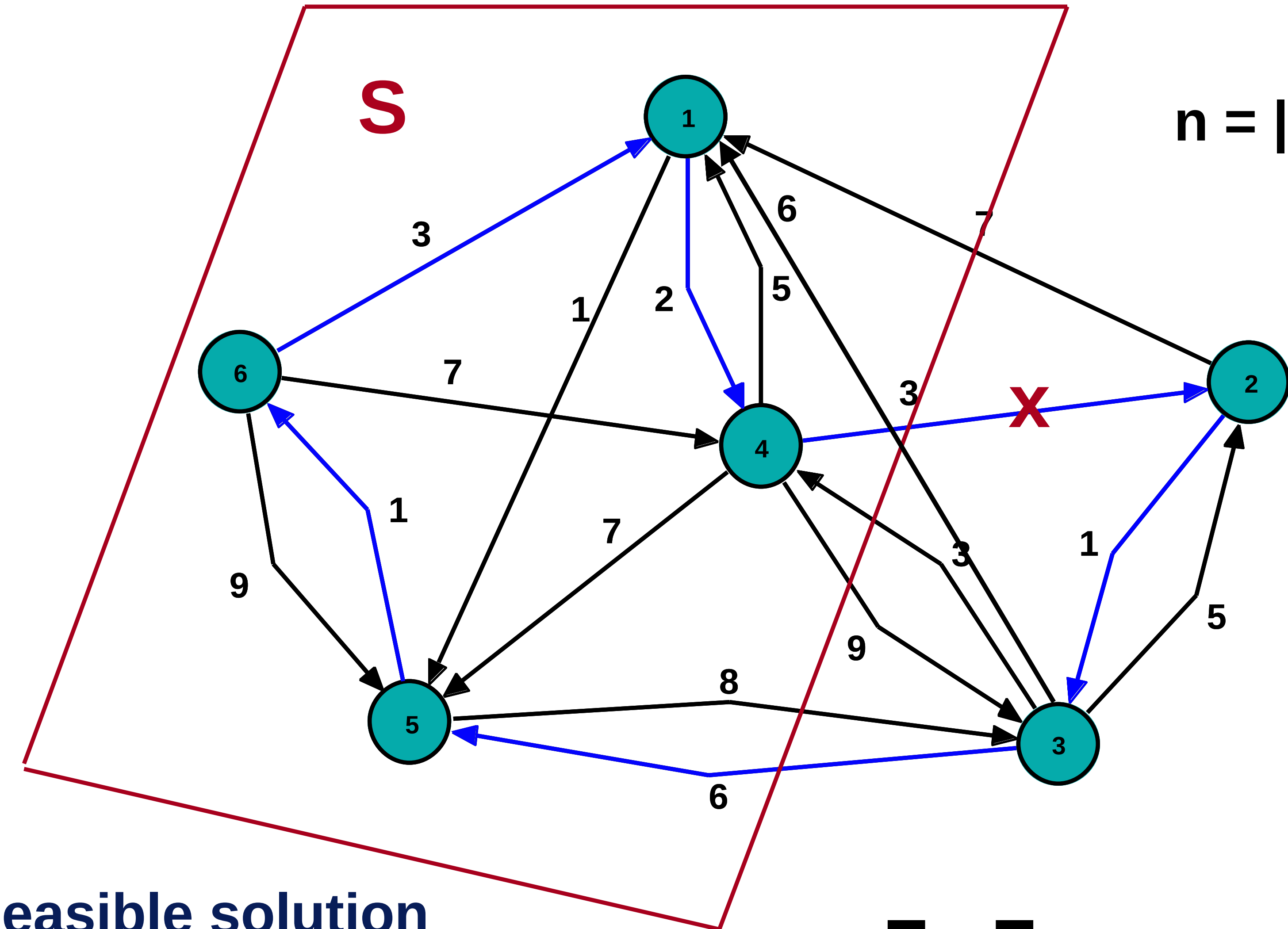


n = |V| = 6

X Infeasible solution

$$\sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \geq 1$$ NO!

Example



n = |V| = 6

Feasible solution

$$\sum_{i\in S}\ \sum_{j\in V\setminus S} x_{ij} \geq 1$$ **YES!**

LP LOWER BOUND

- The value of the optimal solution of the Linear Programming (LP) Relaxation (or Continuous Relaxation) of the previous formulations, obtained by replacing

$x_{ij} \in \{0, 1\}$ $\quad\quad$ $i \in V, j \in V$

with

$0 \leq x_{ij} \leq 1$ $\quad\quad$ $i \in V, j \in V$

represents a valid Lower Bound on the value of the optimal solution of the ATSP.

- This LP Relaxation can be efficiently (polynomially) solved by using appropriate Separation Procedures (Polynomial Separation Problem).

- The LP Relaxation can be strengthened (so as to obtain better lower bounds) by adding valid inequalities, which are "redundant" for the ILP model, but can be violated by its LP Relaxation (exact and/or heuristic *"separation procedures"*).

ALTERNATIVE ILP FORMULATIONS

- Miller-Tucker-Zemlin (J. ACM, 1960);
- Fox-Gavish-Graves (Operations Research 1980);
- Wong (IEEE Conference …, 1980);
- Claus (SIAM J. on Algebraic Discrete Methods, 1984);
- Finke-Claus-Gunn (Congressus Numerantium, 1984);
- Langevin-Soumis-Desrosiers (Operations Research Letters, 1990)
- Desrochers-Laporte (Operations Research Letters, 1991);
- Gouveia-Voss (European Journal of Operational Research, 1995)
- Gouveia-Pires (European Journal of Operational Research, 1999);
- Myung (International Journal of Management, 2001);
- Gouveia-Pires (Discrete Applied Mathematics, 2001);
- Sherali-Driscoll (Operations Research 2002);
- Sarin-Sherali-Bhootra (Oper. Res. Letters, 2005);
- Sherali-Sarin-Tsai (Discrete Optimization, 2006);
- Oncan-Altinel-Laporte (Review, Computers & Operations Research, 2009)

ALTERNATIVE ILP FORMULATIONS

- These formulations involve a polynomial number of constraints ("compact formulations"),

  but

- their Linear Programming Relaxations generally produce Lower Bounds weaker (and more time consuming) than those corresponding to the Dantzig-Fulkerson-Johnson formulation (with the addition of valid inequalities).

Hamiltonian Circuit Problem (*HC*)

- Given a (directed or undirected) graph $G = (V,A)$ with:
  - $V = \{1, \dots, u\}$ vertex set
  - $A = \{(i, j)\}$ arc set, with $m = |A|$, $m \leq u * u$, $u \leq m$

    Arc h: (i_h, j_h) $h = 1, \dots, m$
- *HC*: determine if a HAMILTONIAN CIRCUIT exists in G.

  * *HC* is known to be NP-Hard

ATSP is NP-hard

- **Input: n, (c_{ij}) : size 1 + n^2 : n^2**

- **Binary decision tree with n^2 levels (variables x_{ij})**

- ***ATSP* $\in$ *Class NP***

ATSP is NP-Hard

ATSP ∈ *Class NP*

HC ∝ *ATSP*

Given any instance of *HC*: *u, m, A* (Size: *u * u*)

1) **Define (in time $O(u * u)$) an instance $(n, (c_{ij}))$ of *ATSP*:**

 * $n := u$

 * $c_{ij} := 0$ if $(i, j) \in A$, $c_{ij} := 1$ otherwise ($i = 1, .., n; j = 1, \ldots, n$).

2) **Determine the optimal solution (x_{ij}, z) of *ATSP*.**

3) **If $z = 0$: *HC* has a feasible solution (x_{ij})**

   If $z \geq 1$: *HC* has a no feasible solution

Computing time $O(n*n)$ (hence $O(u*u)$, polynomial in the size of *HC*)

Shortest Spanning Arborescence with root r (SSA(r))

- **Given a complete DIRECTED GRAPH G = (V,A) with:**
  - **V = {1, …, n} vertex set; A arc set; *r* ∈ V;**
  - **c_{ij} = cost associated with arc (*i*, *j*) ∈ A ($c_{ii} = \infty$, *i* ∈ V).**

***Spanning Arborescence* with root in vertex *r*:**

a) (*n* – 1) arcs;

b) “Connected” with respect to vertex *r*;

c) “Acyclic” (with no circuit).

***SSA(r)*: Find a *Spanning Arborescence* with root *r* whose global cost is minimum.**

***SSA(r)* is a polynomial problem (Edmonds alg: (O(n^2))**

INTEGER LINEAR PROGRAMMING FORMULATION

$x_{ij} = 1$ if arc (i, j) is in the optimal solution

$x_{ij} = 0$ otherwise $\quad i \in V,\ j \in V$

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij}\, x_{ij}$$

s.t.

$$\sum_{i \in V} \sum_{j \in V} x_{ij} = n - 1$$

$$\sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \geq 1 \qquad S \subset V,\ r \in S$$

$$x_{ij} \in \{0, 1\} \qquad i \in V, j \in V$$

CONNECTIVITY CONSTRAINTS

(impose the connectivity of the solution; $O(2^n)$)

INTEGER LINEAR PROGRAMMING FORMULATION

$x_{ij} = 1$ if arc (i, j) is in the optimal solution

$x_{ij} = 0$ otherwise $\quad i \in V,\ j \in V$

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij}\, x_{ij}$$

s.t.

$$\sum_{i \in V} \sum_{j \in V} x_{ij} = n - 1 \quad \text{(redundant if } c_{ij} > 0\text{)}$$

$$\sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \geq 1 \qquad S \subset V,\ r \in S$$

CONNECTIVITY CONSTRAINTS

(impose the connectivity of the solution; $O(2^n)$)

$$x_{ij} \in \{0, 1\} \qquad i \in V, j \in V$$