The diet problem

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Summary: The goal of the diet problem is to select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost. The problem is formulated as a linear program where the objective is to minimize cost and the constraints are to satisfy the specified nutritional requirements. The diet problem constraints typically regulate the number of calories and the amount of vitamins, minerals, fats, sodium, and cholesterol in the diet. While the mathematical formulation is simple, the solution may not be palatable! The nutritional requirements can be met without regard for taste or variety, so consider the output before digging into a meal from an optimal menu!

History: The diet problem was one of the first optimization problems studied in the 1930s and 1940s. The problem was motivated by the Army's desire to minimize the cost of feeding GIs in the field while still providing a healthy diet. One of the early researchers to study the problem was George Stigler, who made an educated guess of an optimal solution using a heuristic method. His guess for the cost of an optimal diet was \$39.93 per year (1939 prices). In the fall of 1947, Jack Laderman of the Mathematical Tables Project of the National Bureau of Standards used the newly developed simplex method to solve Stigler's model. As the first "large scale" computation in optimization, the linear program consisted of nine equations in 77 unknowns. It took nine clerks using hand-operated desk calculators 120 man days to solve for the optimal solution of \$39.69. Stigler's guess was off by only \$0.24 per year!

Problem statement: Given a set of foods, along with the nutrient information for each food and the cost per serving of each food, the objective of the diet problem is to select the number of servings of each food to purchase (and consume) so as to minimize the cost of the food while meeting the specified nutritional requirements. Typically, the nutritional requirements are expressed as a minimum and a maximum allowable level for each nutritional component. Other constraints such a minimum and/or maximum number of servings may be included to improve the quality of the menu.

Notation:
Set of foods: F, set of nutrients: N.
Amount of nutrient $j \in N$ present in food $i \in F$: a_{ij}.
Cost for one serving of food $i \in F$: c_i.
Minimum and maximum number of servings of food $i \in F$: m_i and M_i, respectively. Minimum and maximum quantity of nutrient $j \in N$: l_j and L_j, respectively.

Formulation: We solve this problem using a single variable set: $x_i \in \mathbb{N}$ being the number of servings of food $i \in F$ that we want to purchase and serve. A model for the problem is:

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\begin{aligned}
\min \quad & \sum_{i \in F} c_i x_i & & (1)\\
\text{s.t.} \quad & \sum_{i \in F} a_{ij} x_i \geq l_j & \forall j \in N \quad & (2)\\
& \sum_{i \in F} a_{ij} x_i \leq L_j & \forall j \in N \quad & (3)\\
& m_i \leq x_i \leq M_i & \forall i \in F \quad & (4)\\
& x_i \in \mathbb{N} & \forall i \in F \quad & (5)
\end{aligned}
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