Online appendix
Recent Developments in Location Routing Problems
Deterministic multi-period, multi-echelon, multi-objective problems

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A. Mathematical models of selected location-routing problem variants

In this section, we present mathematical models for what we consider standard variants of a periodic and a two-echelon location-routing problem (LRP).

A.1. Periodic location-routing problem

We present a four-index formulation for the periodic LRP (PLRP, see Section 2.1 in the survey) based on the model of Prodhon (2011). Table 1 summarizes the notation used in the following mathematical formulation.

A planning period H composed of several days is considered. The problem is defined on a complete and undirected graph $G = (V, E)$. The set V represents all locations and includes the set of facility locations I and the set of customer locations J. The set E denotes the edges linking any two locations i and j. The symmetric cost for traveling from node $i \in V$ to $j \in V \setminus \{i\}$ is represented by c_{ij}. Each facility location $i \in I$ has an opening cost O_i and a capacity of W_i units. Each customer $j \in J$ has to be served a given number of times s_j, C_j is the set of allowed visiting patterns for customer $j \in J$, and the binary parameter a_{rl} is equal to one if day $l \in H$ belongs to visiting pattern $r \in C_j$. Moreover, the demand of customer $j \in J$ on the day $l \in H$ of visiting pattern $r \in C_j$ is represented by d_{jlr}. A fleet K of N identical vehicles with capacity Q is available over the planning period H. If a vehicle is used at least once from a facility during the horizon, a fixed cost F is charged.

Sets	Definition
I	Set of potential facility locations.
J	Set of customer locations.
$V = I \cup J$	Set of all locations.
K	Set of vehicles.
H	Set of days (planning period).
C_j	Set of visiting patterns for customer $j \in J$.

Parameters	Definition
c_{ij}	Cost for traveling from location $i \in V$ to $j \in V \setminus \{i\}$.
O_i	Opening cost of facility $i \in I$.
W_i	Capacity of facility $i \in I$.
F	Vehicle fixed cost.
Q	Vehicle capacity.
N	Total number of available vehicles.
s_j	Visit frequency for customer $j \in J$.
d_{jlr}	Demand of customer $j \in J$ on day $l \in H$ of visiting pattern $r \in C_j$.
a_{rt}	One if day t belongs to visiting pattern $r \in C_j$.

Variables	Definition
$x_{ijkl} \in \{0,1\}$	One if vehicle $k \in K$ travels directly from location $i \in V$ to $j \in V \setminus \{i\}$ on day $l \in H$, zero otherwise.
$w_{ij} \in \{0,1\}$	One if customer $j \in J$ is served by facility $i \in I$, zero otherwise.
$y_i \in \{0,1\}$	One if facility $i \in I$ is opened, zero otherwise.
$N_i \in \mathbb{N}$	Total number of vehicles used at facility $i \in I$.
$b_{jr} \in \{0,1\}$	One if visiting pattern $r \in C_j$ is assigned to customer $j \in J$.

Table 1: Summary of the mathematical notation used in Section A.1.

In addition, each vehicle can operate at most one single route per day. Thus, the total number of vehicles N_i used at facility i is a decision variable that corresponds to the maximum number of routes performed during any day from facility i over H. The binary variable x_{ijkl} is equal to one if vehicle $k \in K$ travels from node $i \in V$ to $j \in V \setminus \{i\}$ on day $l \in H$ and zero otherwise. Moreover, we introduce the binary variable w_{ij}, which is equal to one if customer $j \in J$ is served by facility $i \in I$ and zero otherwise. The binary variable y_i indicates whether facility $i \in I$ is opened or not. Finally, the binary variable b_{jr} is equal to one if visiting pattern $r \in C_j$ is assigned to customer $j \in J$.

Using the above notation, the PLRP can be formulated as follows:

$$\min \quad \sum_{i \in V} \sum_{j \in V \setminus \{i\}} \sum_{k \in K} \sum_{l \in H} x_{ijkl} c_{ij} + \sum_{i \in I} N_i F + \sum_{i \in I} y_i O_i \tag{1a}$$

$$\text{subject to} \quad \sum_{k \in K} \sum_{j \in J} x_{ijkl} \leq N_i \qquad\qquad \forall l \in H, \forall i \in I \tag{1b}$$

$$\sum_{j \in V \setminus \{i\}} x_{ijkl} - \sum_{h \in V \setminus \{i\}} x_{hikl} = 0 \qquad\qquad \forall k \in K, \forall i \in V, \forall l \in H \tag{1c}$$

$$\sum_{i \in S} \sum_{j \in S \setminus \{i\}} x_{ijkl} \leq |S| - 1 \qquad\qquad \forall S \subseteq J, \forall k \in K, \forall l \in H \tag{1d}$$

$$\sum_{j \in J} \sum_{i \in V \setminus \{j\}} d_{jlr} x_{ijkl} \leq Q \qquad\qquad \forall k \in K, \forall l \in H, \forall r \in C_j \tag{1e}$$

$$\sum_{j \in J} w_{ij} d_{jlr} \leq W_i y_i \qquad\qquad \forall i \in I, \forall l \in H, \forall r \in C_j \tag{1f}$$

$$\sum_{i \in I} \sum_{j \in J} x_{ijkl} \leq 1 \qquad\qquad \forall k \in K, \forall l \in H \tag{1g}$$

$$\sum_{u \in J} x_{iukl} + \sum_{u \in V \setminus \{j\}} x_{ujkl} \leq 1 + w_{ij} \qquad\qquad \forall i \in I, \forall j \in J, \forall k \in K, \forall l \in H \tag{1h}$$

$$\sum_{r \in C_j} b_{jr} = 1 \qquad\qquad \forall j \in J \tag{1i}$$

$$\sum_{i \in V \setminus \{j\}} \sum_{k \in K} x_{ijkl} - \sum_{r \in C_j} b_{jr} a_{rl} = 0 \qquad\qquad \forall j \in J, l \in H \tag{1j}$$

$$x_{ijkl} \in \{0,1\} \qquad\qquad \forall i \in V, \ \forall j \in V \setminus \{i\}, k \in K, l \in H \tag{1k}$$

$$w_{ij} \in \{0,1\} \qquad\qquad \forall i \in I, \ \forall j \in J \tag{1l}$$

$$y_i \in \{0,1\} \qquad\qquad \forall i \in I \tag{1m}$$

$$b_{jr} \in \{0,1\} \qquad\qquad \forall j \in J, \forall r \in C_j \tag{1n}$$

$$N_i \in \mathbb{N} \qquad\qquad\qquad\qquad \forall i \in I. \qquad\qquad (1\text{o})$$

The objective function (1a) minimizes the sum of total routing cost, vehicle fixed cost, and facility opening cost. Constraints (1b) ensure that the number of vehicles departing from each facility on each day of the planning horizon does not exceed the maximum number of vehicles assigned to it. Constraints (1c) guarantee that if a vehicle visits a location then it also leaves it. Constraints (1d) are the subtour elimination constraints. Constraints (1e) and (1f) are the capacity constraints for the vehicles and the facilities, respectively. Constraints (1g) ensure that each vehicle can depart at most once from each facility, on each day of the planning horizon. Constraints (1h) specify that a customer can be assigned to a facility only if a route linking them to the facility is performed. Constraints (1i) guarantee that exactly one visiting pattern is assigned to each customer, and constraints (1j) ensure that each customer is visited exactly once on each day of the assigned visiting pattern. Finally, constraints (1k)–(1o) define the domain of the variables.

A.2. Two-echelon location-routing problem

In this section, we present a formulation for a two-echelon location-routing problem (2ELRP) in which goods are transported on multi-stop routes on both echelons. This problem involves a single depot with a fixed location, and location decisions only affect the facilities. The formulation is based on the one presented by Nguyen et al. (2012), which uses an undirected graph $G = (\bar{V}, E)$. Refer to Table 2 for the complete notation. The problem is presented in a distribution setting. Therefore, the first echelon is the one farther away from the customers (linking the depot to the facilities), and the second echelon is the one closer to the customers (linking facilities to customers). In the model, we use the following quantities:

- A lower bound on the number of first-echelon vehicles necessary to serve facilities $U \subseteq I$,

$$r_1(U) = \frac{1}{Q_1} \sum_{i \in U} \sum_{j \in J} d_j w_{ij}. \qquad\qquad (2)$$

Note that r_1 involves variables w_{ij} and, thus, is not a constant.

- A lower bound on the number of second-echelon vehicles necessary to serve customers $U \subseteq J$,

$$r_2(U) = \left\lceil \frac{1}{Q_2} \sum_{j \in U} d_j \right\rceil. \qquad\qquad (3)$$

Note that r_2 only involves the problem parameters and, therefore, is a constant.

Sets	Definition
$\{0\}$	Central depot singleton.
I	Set of potential facility locations.
J	Set of customer locations.
$\bar{V} = \{0\} \cup I \cup J$	Set of all locations.
$V' = \{0\} \cup I$	Set of locations used in the first echelon.
$V = I \cup J$	Set of locations used in the second echelon.
E	Set of undirected edges.
$\delta(U) \subseteq E$	Set of edges with exactly one endpoint in $U \subseteq \bar{V}$.
$\delta(u) = \delta(\{u\})$	Short notation for edges incident to $u \in \bar{V}$.
$\gamma(U) \subseteq E$	Set of edges with both endpoints in $U \subseteq \bar{V}$.
$E(U, U') \subseteq E$	Set of edges with exactly one endpoint in $U \subseteq \bar{V}$ and the other in $U' \subseteq \bar{V}$ ($U \cap U' = \emptyset$).
$E_1 = \gamma(V')$	Set of first-echelon edges.
$E_2 = \gamma(V)$	Set of second-echelon edges.

Parameters	Definition
c_{uv}	Travel cost between $u \in \bar{V}$ and $v \in \bar{V}$.
W_i	Capacity of a facility opened at location $i \in I$.
O_i	Fixed cost incurred if opening a facility at location $i \in I$.
d_j	Demand of customer $j \in J$.
Q_1	First-echelon vehicle capacity.
Q_2	Second-echelon vehicle capacity.
F_1	First-echelon vehicle fixed cost.
F_2	Second-echcelon vehicle fixed cost.

Variables	Definition
$w_{ij} \in \{0, 1\}$	Takes the value one if and only if customer $j \in J$ is assigned to a facility opened at location $i \in I$.
$x^1_{uv} \in \{0, 1, 2\}$	Number of times edge $uv \in E_1$ is traversed.
$x^2_{uv} \in \{0, 1, 2\}$	Number of times edge $uv \in E_2$ is traversed.
$y_i \in \{0, 1\}$	Takes the value one if and only if a facility is opened at location $i \in I$.

Table 2: Summary of the mathematical notation used in Section A.2.

The integer formulation is as follows:

$$\min \quad \sum_{i \in I} O_i y_i +$$

$$\sum_{i \in I} \frac{F_1}{2} x_{0i}^1 + \sum_{i \in I} \sum_{j \in J} \frac{F_2}{2} x_{ij}^2 +$$

$$\sum_{uv \in E_1} c_{uv} x_{uv}^1 + \sum_{uv \in E_2} c_{uv} x_{uv}^2 \tag{4a}$$

$$\text{subject to} \quad \sum_{iu \in \delta(i)} x_{iu}^1 = 2y_i \qquad \forall i \in I \tag{4b}$$

$$\sum_{uv \in \delta(U)} x_{uv}^1 \geq 2r_1(U) \qquad \forall U \subseteq I \tag{4c}$$

$$\sum_{ju \in \delta(j)} x_{ju}^2 = 2 \qquad \forall j \in J \tag{4d}$$

$$\sum_{uv \in \delta(U)} x_{uv}^2 \geq 2r_2(U) \qquad \forall U \subseteq J \tag{4e}$$

$$\sum_{i \in U} x_{ij}^2 + 2 \sum_{uv \in \gamma(U' \cup \{j,l\})} x_{uv}^2 + \sum_{i \in I \setminus U} x_{il}^2 \leq 2|U'| + 3 \qquad \begin{array}{l} \forall j, l \in J, \ \forall U \subseteq I, \\ \forall U' \subseteq J \setminus \{j,l\}, \\ U' \neq \emptyset \end{array} \tag{4f}$$

$$\sum_{i \in U} x_{ij}^2 + 3x_{jl}^2 + \sum_{i \in I \setminus U} x_{il} \leq 4 \qquad \forall j, l \in J, \ \forall U \subseteq I \tag{4g}$$

$$\sum_{j \in J} d_j w_{ij} \leq W_i y_i \qquad \forall i \in I \tag{4h}$$

$$\sum_{u \in V \setminus (U \cup U')} x_{ij}^2 + r_2(U') \sum_{uv \in E(U, V' \setminus U)} x_{uv}^1 \geq 2r_2(U') \qquad \forall U \subseteq I, \ \forall U' \subseteq J \tag{4i}$$

$$x_{uv}^1 \in \{0,1\} \qquad \forall uv \in \gamma(I) \tag{4j}$$

$$x_{uv}^1 \in \{0,1,2\} \qquad \forall uv \in \delta(0) \tag{4k}$$

$$x_{uv}^2 \in \{0,1\} \qquad \forall uv \in \gamma(J) \tag{4l}$$

$$x_{uv}^2 \in \{0,1,2\} \qquad \forall uv \in E(I,J) \tag{4m}$$

$$y_i \in \{0,1\} \qquad \forall i \in I \tag{4n}$$

$$w_{ij}\{0,1\} \qquad \forall i \in I, \ \forall j \in J. \tag{4o}$$

The objective function (4a) minimizes the sum of the facility opening costs, the vehicle fixed costs of both first- and second-echelon vehicles, and the travel cost on both echelons. Constraints (4b) impose that a facility is visited by exactly one first-echelon route if it is open, and by none if it is closed. Analogously, constraints (4d) impose that each customer is visited exactly once. Constraints (4c) and

(4e) prevent subtours in the first and second echelons, respectively, ensuring that a minimum number of vehicles visit each subset of facilities and customers. Constraints (4f) and (4g) are path-elimination constraints forbidding paths starting at one facility and ending at another. They forbid a customer from being connected to two different facilities because this would imply that a facility-to-facility path exists. We remark that these constraints are often used in formulations of other LRPs (see, e.g., Belenguer et al., 2011). Constraints (4h) are facility capacity constraints. Constraints (4i) ensure that any subset of customers is connected with the main depot through a subset of facilities. They are adapted from constraints introduced for a related problem by Belenguer et al. (2016).

References

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