Application of Dantzig-Wolfe Reformulation to Binary Quadratic Problems

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Plan

Context

- In this work we aim at investigating a family of relaxations for Binary Quadratic Problems (BQPs).
- A generic BQP reads as follows :

Binary Quadratic Problem (BQP)

$$(BQP) \quad \max\{f(x) = x^\top Qx + L^\top x | Ax \leq b, x \in \{0,1\}\} \; .$$

- $Q \in \mathbb{Q}^{n \times n}$ and not restricted to be convex.
- $L \in \mathbb{Q}^n$.

Formulation

Notations

n : number of items
a_j : weight of item j ($j = 1, \ldots, n$)
b : capacity of the knapsack
c_{ij} : profit associated with the selection of items i and j ($i, j = 1, \ldots, n$)
k : number of items to be filled in the knapsack

Assumptions

$c_{ij} \in \mathbb{N}\ i, j = 1, \ldots, n,\ a_j \in \mathbb{N}\ j = 1, \ldots, n,\ b \in \mathbb{N}$
$\max_{j=1,\ldots,n} a_j \leq b < \sum_{j=1}^{n} a_j$
$k \in \{1, \ldots, k_{max}\}$

Formulation

k-item Quadratic Knapsack Problem (kQKP)

$$(kQKP) \quad \max \quad f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j \qquad (1)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_j x_j \leq b \qquad (2)$$

$$\sum_{j=1}^{n} x_j = k \qquad (3)$$

$$x_j \in \{0, 1\} \qquad j = 1, \ldots, n \qquad (4)$$

Complexity

NP-hard problem : generalization of (QKP) and k-cluster problems

Plan

Reformulations for (kQKP)

Reformulation of the objective function $f(x)$

QCR/MQCR Method

($f(x)$ can be reformulated by exploiting the property $x^2 = x$)

$\Downarrow$

convex problem

Reformulation of the feasible region $\{x|Ax \leq b, x \in \{0,1\}\}$

Danzig-Wolfe Reformulation

(a subset of constraints is substituted by its convex hull)

$\Downarrow$

tighter formulation

Quadratic Convex Reformulation (QCR) : a two-phase method

[Billionnet, Elloumi and M-C. Plateau, Discrete Applied Mathematics, 2009]

- **Phase 1** : Reformulate the objective function $f(x)$ into an equivalent 0-1 program with a **concave** quadratic objective function $f_{u,\alpha}(x)$.
  $\rightarrow$ an equivalent **convex** 0-1 program
- **Phase 2** : Apply a standard 0-1 convex quadratic solver to this new problem.

QCR - Phase I : Addition of two functions null on the feasible set

- $q_u(x) = \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right)$
- $q_\alpha(x) = \sum_{i=1}^{n} \alpha_i x_i \left(\sum_{j=1}^{n} x_j - k\right)$ or $q_v(x) = v\left(\sum_{j=1}^{n} x_j - k\right)^2$

q_α and q_v are implied by the cardinality constraint.

QCR : A two-phase method

QCR - Phase I : Reformulation 1

New objective function (concave iff quadratic terms matrix is SDP) :

$$f_{u,\alpha}(x) = f(x) - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - \sum_{i=1}^{n} \alpha_i x_i \left(\sum_{j=1}^{n} x_j - k\right)$$

Reformulation 2 [Faye, Roupin, 4OR, 2007 ; Billionnet, Elloumi and Lambert, Math. Prog., 2012]

$$f_{u,v}(x) = f(x) - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - v \left(\sum_{j=1}^{n} x_j - k\right)^2$$

- identical bounds.
- consumes the least amount of computation times.

QCR : A two-phase method

Best values for the parameters $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$

Solve the semi-definite relaxation of the problem $(E - kQKP)$:

$$E-kQKP_{(SDP1)} \left\{ \begin{array}{lll} \max & \sum_{i=1}^n \sum_{j=1}^n c_{ij} X_{ij} & \\ \text{s.t.} & X_{ii} = x_i \qquad i = 1, \ldots, n & (u^*) \\ & \sum_{i=1}^n \sum_{j=1}^n X_{ij} - 2k \sum_{j=1}^n x_j = -k^2 & (v^*) \\ & \sum_{j=1}^n a_j x_j \leq b & \\ & \sum_{j=1}^n x_j = k & \\ & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 & \\ & x \in \mathbb{R}^n,\ X \in \mathbb{R}^{n \times n} & \end{array} \right.$$

An improved convex 0-1 quadratic reformulation for $(E-kQKP)$ (1/4) [Billionnet, Elloumi and Lambert, Math. Prog., 2012 and Ji, Zheng and Sun, Pacific Journal of Optimization, 2012]

From

$$f(x) = x^t Cx$$

Consider the decomposition of $f(x)$

$$f(x) = x^t(C-M)x + x^t Mx,$$

where

$$C - M \preceq 0$$

and

$$M = Diag(u) + P + N$$

with $u \in \mathbb{R}^n$, $P \in \mathbb{R}_+^{n\times n}$ and $N \in \mathbb{R}_-^{n\times n}$

An improved convex 0-1 quadratic reformulation for $(E-kQKP)$ (2/4)

$$f(x) = x^t(C - Diag(u) - P - N)x + x^t(Diag(u) + P + N)x$$

$$= x^t(C - Diag(u) - P - N)x + u^t x + \sum_{i,j=1}^{n}[P_{ij}x_ix_j + N_{ij}x_ix_j]$$

$$= x^t(C - Diag(u) - P - N)x + u^t x + \sum_{i,j=1}^{n}[P_{ij}s_{ij} - N_{ij}t_{ij}]$$

where

$s_{ij} = min(x_i, x_j)$,

$t_{ij} = -max(0, x_i + x_j - 1)$

($x_ix_j = min(x_i, x_j) = max(0, x_i + x_j - 1)$ for any $x_i, x_j \in \{0, 1\}$)

An improved convex 0-1 quadratic reformulation for $(E-kQKP)$ (3/4)

Relax

$s_{ij} = min(x_i, x_j)$ to two linear inequalities $s_{ij} \leq x_i$ and $s_{ij} \leq x_j$

and

$t_{ij} = -max(0, x_i + x_j - 1)$ to $t_{ij} \leq 0$ and $t_{ij} \leq 1 - x_i - x_j$.

without affecting the optimal solution of the problem.

The reformulation is equivalent to the convex 0-1 quadratic program :

$$\begin{cases} \max & f(x) = x^t(C - Diag(u) - P - N)x + u^t x + \sum_{i,j=1}^{n} [P_{ij} s_{ij} + N_{ij} t_{ij}] \\ \text{s.t.} & \sum_{j=1}^{n} a_j x_j \leq b \\ & \sum_{j=1}^{n} x_j = k \\ & s_{ij} \leq x_i,\ s_{ij} \leq x_j \quad i,j = 1, \ldots, n \\ & t_{ij} \leq 0,\ t_{ij} \leq 1 - x_i - x_j \quad i,j = 1, \ldots, n \\ & x_j \in \{0, 1\} \quad j = 1, \ldots, n \end{cases}$$

An improved convex 0-1 quadratic reformulation for $(E - kQKP)$ (4/4)

The optimal parameters (u^*, v^*, P^*, N^*) can be found by solving a SDP problem :

$$
(E - kQKP_{SDP2}) \left\{
\begin{array}{llll}
\max & \sum_{i=1}^n \sum_{j=1}^n c_{ij} X_{ij} & & \\
\text{s.t.} & X_{ii} = x_i & i = 1, \dots, n & (u^*) \\
& \sum_{i=1}^n \sum_{j=1}^n X_{ij} - 2k \sum_{j=1}^n x_j = -k^2 & i = 1, \dots, n & (v^*) \\
& \sum_{j=1}^n a_j x_j \le b & & \\
& \sum_{j=1}^n x_j = k & & \\
& X_{ij} \le x_i, \ X_{ij} \le x_j & i,j = 1, \dots, n & (P^*) \\
& X_{ij} \ge x_i + x_j - 1, \ X_{ij} \ge 0 & i,j = 1, \dots, n & (N^*) \\
& \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 & & \\
& x \in \mathbb{R}^n, \ X \in S_n & &
\end{array}
\right.
$$

How to manage the $4n^2$ bound constraints

(1) $X_{ij} \leq x_i \; i,j = 1,\ldots,n$; (2) $X_{ij} \leq x_j \; i,j = 1,\ldots,n$
(3) $X_{ij} \geq x_i + x_j - 1 \; i,j = 1,\ldots,n$; (4) $X_{ij} \geq 0 \; i,j = 1,\ldots,n$

- *IQCR* : Ji, Zheng and Sun propose to keep only the constraints such that $|\, i - j \,| \leq 5$
- *MQCR* (our iterative method) :

  $V \leftarrow \phi$

  Repeat

  Solve the SDP problem with constraints of V /**QCR* at 1st iteration*/

  Detect the violated constraints of (1) such that $X_{ij} > x_i + 10^{-4}$

  Keep the set V of the most violated constraints (at most $1000 + 15n$)

  Until the number of violated constraints is less than 30

- constraints (2) : transformed into constraints (1) thanks to the symmetry of X

- constraints (3) and (4) : small number of violated constraints ; weak improvement of the bound

The application of the QCR method leads to the following reformulation of $(kQKP)$:

$$(kQKP^{v^*,u^*}) \quad \max \; f_{v^*,u^*}(x) = f(x) - \sum_{i=1}^{n} u_i^* \left(x_i^2 - x_i\right) - v^* \left(\sum_{j=1}^{n} x_j - k\right)^2 \tag{5}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_j x_j \leq b \tag{6}$$

$$\sum_{j=1}^{n} x_j = k \tag{7}$$

$$x \in \{0,1\} \tag{8}$$

The application of the MQCR method leads to the following reformulation of $(kQKP)$:

$$
\begin{aligned}
(kQKP_M^{u^*,v^*,P^*,N^*}) \quad \max \quad & f_{u^*,v^*,P^*,N^*}(x,s,t) \\
& = x^T(C - Diag(u) - P - N)x + u^T x + \sum_{i,j=1}^{n} [P_{ij}s_{ij} + N_{ij}t_{ij}] \\
& - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - v\left(\sum_{j=1}^{n} x_j - k\right)^2 \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \leq b \\
& \sum_{j=1}^{n} x_j = k \\
& s_{ij} \leq x_i, \; s_{ij} \leq x_j \qquad i,j = 1,\ldots,n \\
& t_{ij} \leq 0, \; t_{ij} \leq 1 - x_i - x_j \qquad i,j = 1,\ldots,n \\
& x_j \in \{0,1\} \qquad j = 1,\ldots,n
\end{aligned}
$$

Plan

Dantzig Wolfe Reformulation (DWR) for a generic (BQP)

Binary Quadratic Problem (BQP)

$$(BQP) \quad \max\{f(x) = x^\top Qx + L^\top x | Ax \leq b, x \in \{0,1\}\} \; .$$

- Let A', A'' and b', b'' be a generic row partition of the constraint matrix A and of the rhs vector b.
- (BQP) can be strengthened by convexifying the constraints $A''x \leq b''$ (i.e. imposing $x \in conv\{A''x \leq b'''', x \in \{0,1\}\}$)

DWR with quadratic master problem

DWR of constraints A''

(BQP) can be reformulated as follows :

$$
\begin{aligned}
(BQP_{DWR(A'')}) \quad \max \quad & f_{v^*,u^*}(x) && && (9)\\
\text{s.t.} \quad & A'x \leq b' && [\alpha] && (10)\\
& x_j = \sum_{p \in \mathcal{P}_{DWR(A'')}} x_j^p \lambda^p \quad j = 1, \ldots, n && [\tau_j] && (11)\\
& \sum_{p \in \mathcal{P}_{DWR(A'')}} \lambda^p = 1 && [\beta] && (12)\\
& x_j \in \{0, 1\} \quad j = 1, \ldots, n && && (13)\\
& \lambda^p \geq 0 \quad p \in \mathcal{P}_{DWR(A'')} && && (14)
\end{aligned}
$$

with $\mathcal{P}_{DWR(A'')}$ being the set of extreme points of $conv\{x|A''x \leq b'''', x \in \{0, 1\}\}$.

- Variables λ are partially enumerated by solving an additional *pricing* problem.

Let α, τ and β being the dual variables associated to the constraints in the continuous relaxation of $(BQP_{DWR(A'')})$.

Pricing problem

$$(\Pi_{BQP_{DWR(A'')}}(\tau^*, \beta^*)) \quad \max \quad {\tau^*}^\top x + \beta^* \tag{15}$$

$$\text{s.t.} \quad A''x \leq b'' \tag{16}$$

$$x_j \in \{0, 1\} \qquad j = 1, \dots, n \tag{17}$$

If the optimal value of $(\Pi_{BQP_{DWR(A'')}}(\tau^*, \beta^*))$ is greater than zero, then a column with positive reduced cost is found and added to the master.

- $(BQP_{DWR}) \Rightarrow f(x)$ is quadratic, the pricing problem is (binary) linear.

DWR with quadratic pricing problem

the objective function can be rewritten directly in terms of the λ variables by introducing $f(\lambda) = \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p$ with

$$c_p = f(x_p) = x_p^\top Q x_p + L^\top x_p \ .$$

$\overline{DWR}$ of constraints A''

$$(BQP_{\overline{DWR}(A'')}) \quad \max \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p \qquad (18)$$

$$\text{s.t.} \quad (10) - (14)$$

DWR with quadratic pricing problem

with $f(x) = \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p$, the pricing problem reduces to the following quadratic problem :

Pricing problem

$$
\begin{aligned}
(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad & x^\top Q x + {\tau^*}^\top x + \beta^* && (19)\\
\text{s.t.} \quad & A'' x \leq b'' && (20)\\
& x_j \in \{0, 1\} \qquad j = 1, \ldots, n && (21)
\end{aligned}
$$

where Q is not required to be convex.

- $(BQP_{\overline{DWR}}) \Rightarrow f(\lambda)$ is linear, the pricing problem is (binary) quadratic.

DWR with quadratic pricing problem

The objective function can still be modified with the coefficients u and v :

$$c_p = f_{v,u}(x_p)\ .$$

The pricing reduces to the following quadratic problem :

Pricing problem

$$(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad f_{v,u}(x_p) + {\tau^*}^\top x + \beta^* \tag{22}$$

$$\text{s.t.} \quad A''x \leq b'' \tag{23}$$

$$x_j \in \{0, 1\} \qquad j = 1, \ldots, n \tag{24}$$

where v and u can take any value.

Plan

DWR with a quadratic master problem

Decompositions with a master objective function of the shape $f(x) = x^\top Q x + L^\top x$.

- PRO : linear pricing problem.
- CON : the objective function must be convex. $\Rightarrow$ DWR must be applied to $(kQKP^{v^*,u^*})$ or to $(kQKP_M^{u^*,v^*,P^*,N^*})$

DWR with a quadratic master problem

When applied to $(kQKP^{v^*,u^*})$, DWR gives the following model :

$$
\begin{aligned}
(kQKP^{v^*,u^*}_{DWR(1)}) \quad \max \quad & f_{v^*,u^*}(x) && && (25)\\
\text{s.t.} \quad & \sum_{j=1}^{n} x_j = k && && (26)\\
& x_j = \sum_{p \in \mathcal{P}_{(1)}} x_j^p \lambda^p && j = 1, \ldots, n \quad [\tau_j] && (27)\\
& \sum_{p \in \mathcal{P}_{(1)}} \lambda^p = 1 && [\beta] && (28)\\
& x_j \in \{0, 1\} && j = 1, \ldots, n && (29)\\
& \lambda^p \geq 0 && p \in \mathcal{P}_{(1)} && (30)
\end{aligned}
$$

with $\mathcal{P}_{(1)}$ being the set of extreme points of $conv\{x | \sum_{j=1}^{n} a_j x_j \leq b, x \in \{0, 1\}\}$

DWR with a quadratic master problem

The following reformulations can be obtained :

- $(kQKP^{v^*,u^*}_{DWR(1)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1)})$: knapsack constraint is convexified
- $(kQKP^{v^*,u^*}_{DWR(2)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(2)})$: cardinality constraint is convexified
- $(kQKP^{v^*,u^*}_{DWR(1-2)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1-2)})$: both constraints are convexified

DWR with a quadratic pricing problem

Decompositions with a master objective function of the shape $f(x) = c^\top \lambda$.

- The pricing problem is quadratic.
- No restriction is imposed on the convexity of the objective function of the starting compact model.

DWR with a quadratic pricing problem

When applied to $(kQKP)$, DWR gives the following model :

$$
\begin{aligned}
(kQKP_{\overline{DWR}(1)}) \quad \max \quad & \sum_{p \in \mathcal{P}_{(1)}} c^p \lambda^p && && (31)\\
\text{s.t.} \quad & x_j = \sum_{p \in \mathcal{P}_{(1)}} x_j^p \lambda^p && j = 1, \ldots, n && [\phi_j] \quad (32)\\
& \sum_{j=1}^{n} x_j = k && && [\gamma] \quad (33)\\
& \sum_{p \in \mathcal{P}_{(1)}} \lambda^p = 1 && && [\theta] \quad (34)\\
& x_j \in \{0, 1\} && j = 1, \ldots, n && (35)\\
& \lambda^p \geq 0 && p \in \mathcal{P}_{(1)} && (36)
\end{aligned}
$$

with $\mathcal{P}_{(1)}$ being the set of extreme points of $conv\{x | \sum_{j=1}^{n} a_j x_j \leq b, x \in \{0, 1\}\}$ and with $c^p = \sum_{j=1}^{n} c_{ij} x_i^p x_j^p$.

DWR with a quadratic pricing problem

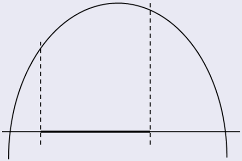
The following reformulations can be obtained :

- $(kQKP_{\overline{DWR}(1)})$, $(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,\overline{DWR}(1)})$: knapsack constraint is convexified
- $(kQKP_{\overline{DWR}(2)})$, $(kQKP^{u^*,v^*}_{\overline{DWR}(2)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,\overline{DWR}(2)})$: cardinality constraint is convexified

Relaxations comparison

Observation

If the problem is convex, the decomposition with a quadratic pricing problem provides a stronger dual bound than a decomposition with a quadratic master problem.



$\Rightarrow$ Modifying the convexity of the quadratic term of the objective function can be crucial.

- for a generic Q, no clear dominance exists between (BQP_{DWR}) and $(BQP_{\overline{DWR}})$.

Plan

Experimental environment

- Carried out on an Intel i7-2600 quad core 3.4 GHz with 8 GB of RAM, using only one core
- CSDP integrated into COIN-OR for solving SDP programs
- CPLEX 12.6 with default settings
- Average values over 10 instances
- $n \in \{50, 60, \ldots, 100\}$
- $k \in [1, n/4]$, $b \in [50, 30k]$, a_j, $c_{ij} \in [1, 100]$

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_M)$ Gap	$(kQKP_M)$ Time	$(kQKP_{DWR(1)})$ Gap	$(kQKP_{DWR(1)})$ Time	$(kQKP_{M,DWR(1-2)})$ Gap	$(kQKP_{M,DWR(1-2)})$ Time
50	*25*	102.65	0.02	30.89	1.05	38.36	1.97		
	50	150.56	0.06	25.25	0.94	31.05	3.04		
	75	230.29	0.12	105.16	1.09	114.26	1.55		
60	*25*	60.76	0.04	130.92	0.04	149.25	1.55		
	50	93.73	0.11	15.08	2.61	19.48	3.86		
	75	212.67	0.25	141.08	2.09	151.22	1.67		
70	*25*	130.23	0.06	38.03	5.11	46.84	4.82		
	50	177.07	0.19	72.81	4.27	80.44	6.83		
	75	382.36	0.44	56.26	3.45	63.77	3.25		
80	*25*	111.24	0.08	34.05	7.98	41.90	5.64		
	50	271.64	0.26	55.44	9.59	64.09	4.67		
	75	313.33	0.66	83.58	7.42	92.31	4.64		
90	*25*	118.45	0.13	112.80	13.75	129.31	4.74		
	50	248.57	0.48	83.15	12.38	92.19	4.52		
	75	388.68	1.06	37.90	5.63	42.13	6.95		
100	*25*	169.43	0.16	73.90	23.49	82.78	6.80		
	50	145.72	0.49	17.38	28.06	21.83	8.58		
	75	260.26	1.25	21.67	18.37	27.22	6.23		
Avg		**198.20**	**0.32**	**63.08**	**8.18**	**71.58**	**4.52**		

TABLE : $(kQKP_{DWR})$, root node, quadratic master

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_M)$ Gap	$(kQKP_M)$ Time	$(kQKP_{DWR(1)})$ Gap	$(kQKP_{DWR(1)})$ Time	$(kQKP_{M,DWR(1-2)})$ Gap	$(kQKP_{M,DWR(1-2)})$ Time
50	*25*	102.65	0.02	30.89	1.05	38.36	1.97	29.15	9.30
	50	150.56	0.06	25.25	0.94	31.05	3.04	23.66	9.71
	75	230.29	0.12	105.16	1.09	114.26	1.55	100.88	8.06
60	*25*	60.76	0.04	130.92	0.04	149.25	1.55	126.07	10.89
	50	93.73	0.11	15.08	2.61	19.48	3.86	14.19	19.05
	75	212.67	0.25	141.08	2.09	151.22	1.67	136.22	8.99
70	*25*	130.23	0.06	38.03	5.11	46.84	4.82	36.52	33.25
	50	177.07	0.19	72.81	4.27	80.44	6.83	70.77	54.37
	75	382.36	0.44	56.26	3.45	63.77	3.25	54.57	22.19
80	*25*	111.24	0.08	34.05	7.98	41.90	5.64	32.87	71.19
	50	271.64	0.26	55.44	9.59	64.09	4.67	53.65	43.98
	75	313.33	0.66	83.58	7.42	92.31	4.64	81.47	43.42
90	*25*	118.45	0.13	112.80	13.75	129.31	4.74	109.63	44.66
	50	248.57	0.48	83.15	12.38	92.19	4.52	81.65	66.75
	75	388.68	1.06	37.90	5.63	42.13	6.95	37.12	102.54
100	*25*	169.43	0.16	73.90	23.49	82.78	6.80	72.72	99.90
	50	145.72	0.49	17.38	28.06	21.83	8.58	17.19	219.77
	75	260.26	1.25	21.67	18.37	27.22	6.23	21.50	158.30
Avg		**198.20**	**0.32**	**63.08**	**8.18**	**71.58**	**4.52**	**61.10**	**57.02**

TABLE : $(kQKP_{DWR})$, root node, quadratic master

Even optimizing over the convex hull of the feasible region does not improve significantly the dual bound provided by DWR with a quadratic master.
The reasons for this behaviour are :

- The optimal solution of the continuous relaxation is attained in the interior of the convex hull of the feasible region.

n	50			60			70			80			90			100		
%	25	50	75	25	50	75	25	50	75	25	50	75	25	50	75	25	50	75
	0	2	0	0	2	0	0	2	0	0	1	0	0	1	0	0	1	0

TABLE : $(kQKP_{DWR})$, Number of instances with optimum of the continuous relaxation in the interior of the convex hull of the feasible region

Even optimizing over the convex hull of the feasible region does not improve significantly the dual bound provided by DWR with a quadratic master.
The reasons for this behaviour are :

- The optimal solution is attained in the interior of the feasible region.

n	50			60			70			80			90			100		
%	25	50	75	25	50	75	25	50	75	25	50	75	25	50	75	25	50	75
	0	2	0	0	2	0	0	2	0	0	1	0	0	1	0	0	1	0

TABLE : $(kQKP_{DWR})$, Number of instances with optimum of the continuous relaxation in the interior of the convex hull of the feasible region

- The objective function over the feasible region is "flat" (due to the convexification via u^* and v^*).

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_{M,DWD(1-2)})$ Gap	$(kQKP_{M,DWD(1-2)})$ Time	$(kQKP_{\overline{DWD}(1)})$ Gap	$(kQKP_{\overline{DWD}(1)})$ Time	$(kQKP^{v^*,u^*}_{\overline{DWD}(1)})$ Gap	$(kQKP^{v^*,u^*}_{\overline{DWD}(1)})$ Time
50	*25*	102.65	0.02	29.15	9.30	19.43	0.30		
	50	150.56	0.06	23.66	9.71	28.97	0.08		
	75	230.29	0.12	100.88	8.06	119.44	0.05		
60	*25*	60.76	0.04	126.07	10.89	79.77	0.14		
	50	93.73	0.11	14.19	19.05	13.13	0.25		
	75	212.67	0.25	136.22	8.99	157.01	0.07		
70	*25*	130.23	0.06	36.52	33.25	16.80	1.20		
	50	177.07	0.19	70.77	54.37	47.85	4.30		
	75	382.36	0.44	54.57	22.19	65.16	0.15		
80	*25*	111.24	0.08	32.87	71.19	11.67	210.59		
	50	271.64	0.26	53.65	43.98	42.69	0.77		
	75	313.33	0.66	81.47	43.42	95.92	14.00		
90	*25*	118.45	0.13	109.63	44.66	57.50	1692.40		
	50	248.57	0.48	81.65	66.75	63.79	190.64		
	75	388.68	1.06	37.12	102.54	59.35	15.16		
100	*25*	169.43	0.16	72.72	99.90	41.07	926.10		
	50	145.72	0.49	17.19	219.77	14.75	577.86		
	75	260.26	1.25	21.50	158.30	47.82	3.77		
Avg		198.20	0.32	61.10	57.02	54.56	202.10		

TABLE : $(kQKP_{DWD})$, root node, quadratic pricer

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_{M,DWD(1-2)})$ Gap	$(kQKP_{M,DWD(1-2)})$ Time	$(kQKP_{\overline{DWD}(1)})$ Gap	$(kQKP_{\overline{DWD}(1)})$ Time	$(kQKP^{v^*,u^*}_{\overline{DWD}(1)})$ Gap	$(kQKP^{v^*,u^*}_{\overline{DWD}(1)})$ Time
50	*25*	102.65	0.02	29.15	9.30	19.43	0.30		
	50	150.56	0.06	23.66	9.71	28.97	0.08		
	75	230.29	0.12	100.88	8.06	119.44	0.05		
60	*25*	60.76	0.04	126.07	10.89	79.77	0.14		
	50	93.73	0.11	14.19	19.05	13.13	0.25		
	75	212.67	0.25	136.22	8.99	157.01	0.07		
70	*25*	130.23	0.06	36.52	33.25	16.80	1.20		
	50	177.07	0.19	70.77	54.37	47.85	4.30		
	75	382.36	0.44	54.57	22.19	65.16	0.15		
80	*25*	111.24	0.08	32.87	71.19	11.67	210.59		
	50	271.64	0.26	53.65	43.98	42.69	0.77		
	75	313.33	0.66	81.47	43.42	95.92	14.00		
90	*25*	118.45	0.13	109.63	44.66	57.50	1692.40		
	50	248.57	0.48	81.65	66.75	63.79	190.64		
	75	388.68	1.06	37.12	102.54	59.35	15.16		
100	*25*	169.43	0.16	72.72	99.90	41.07	926.10		
	50	145.72	0.49	17.19	219.77	14.75	577.86		
	75	260.26	1.25	21.50	158.30	47.82	3.77		
Avg	*25*	115.46	0.08	67.83	44.86	**37.71**	471.79		
	50	181.22	0.26	43.52	68.94	**35.20**	128.98		
	75	297.93	0.63	**71.96**	57.25	90.79	5.53		
	tot	198.20	0.32	61.10	57.02	54.56	202.10		

TABLE : $(kQKP_{DWD})$, root node, quadratic pricer

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_{M,DWD(1-2)})$ Gap	$(kQKP_{M,DWD(1-2)})$ Time	$(kQKP_{\overline{DWD}(1)})$ Gap	$(kQKP_{\overline{DWD}(1)})$ Time	$(kQKP^{v^*,u^*}_{\overline{DWD}(1)})$ Gap	$(kQKP^{v^*,u^*}_{\overline{DWD}(1)})$ Time
50	*25*	102.65	0.02	29.15	9.3	19.43	0.3	0.00	
	50	150.56	0.06	23.66	9.71	28.97	0.08	0.00	
	75	230.29	0.12	100.88	8.06	119.44	0.05	0.00	
60	*25*	60.76	0.04	126.07	10.89	79.77	0.14	0.00	
	50	93.73	0.11	14.19	19.05	13.13	0.25	0.00	
	75	212.67	0.25	136.22	8.99	157.01	0.07	0.00	
70	*25*	130.23	0.06	36.52	33.25	16.8	1.2	0.00	
	50	177.07	0.19	70.77	54.37	47.85	4.3	0.00	
	75	382.36	0.44	54.57	22.19	65.16	0.15	0.00	
80	*25*	111.24	0.08	32.87	71.19	11.67	210.59	0.00	
	50	271.64	0.26	53.65	43.98	42.69	0.77	0.00	
	75	313.33	0.66	81.47	43.42	95.92	14	0.00	
90	*25*	118.45	0.13	109.63	44.66	57.5	1692.4	0.00	
	50	248.57	0.48	81.65	66.75	63.79	190.64	0.00	
	75	388.68	1.06	37.12	102.54	59.35	15.16	0.00	
100	*25*	169.43	0.16	72.72	99.9	41.07	926.1	0.00	
	50	145.72	0.49	17.19	219.77	14.75	577.86	0.00	
	75	260.26	1.25	21.5	158.3	47.82	3.77	0.00	
Avg	*25*	115.46	0.08	67.83	44.86	37.71	471.79	0.00	
	50	181.22	0.26	43.52	68.94	35.2	128.98	0.00	
	75	297.93	0.63	71.96	57.25	90.79	5.53	0.00	
	tot	198.2	0.32	**61.10**	**57.02**	54.56	202.1	**0.00**	

TABLE : $(kQKP_{DWD})$, root node, quadratic pricer

An alternative look at $(kQKP^{v^*,u^*}_{\overline{DWR}(1)})$

Pricing problem

$$
\begin{aligned}
(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad & f_{v,u}(x_p) + \tau^{*\top} x + \beta^* && (37)\\
\text{s.t.} \quad & A'' x \leq b'' && (38)\\
& x_j \in \{0,1\} \qquad j = 1, \ldots, n && (39)
\end{aligned}
$$

$$
f_{v,u}(x_p) = f(x_p) - \sum_{i=1}^{n} u_i \left(x_{pi}^2 - x_{pi}\right) - v \left(\sum_{j=1}^{n} x_j - k\right)^2
$$

$$
f_{v,u}(x_p) = x_p^\top Q x_p + L^\top x_p - v \left(\sum_{j=1}^{n} x_j - k\right)^2
$$

An alternative look at $(kQKP^{v^*,u^*}_{\overline{DWR}(1)})$

Pricing problem

$$(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*,\beta^*)) \quad \max \quad f_{v,u}(x_p) + {\tau^*}^\top x + \beta^* \qquad (40)$$

$$\text{s.t.} \quad A''x \leq b'' \qquad (41)$$

$$x_j \in \{0,1\} \qquad j = 1,\dots,n \qquad (42)$$

$$f_{v,u}(x_p) = f(x_p) - \sum_{i=1}^n u_i \left(x_{pi}^2 - x_{pi}\right) - v\left(\sum_{j=1}^n x_j - k\right)^2$$

$$f_{v,u}(x_p) = x_p^\top Q x_p + L^\top x_p - v\left(\sum_{j=1}^n x_j - k\right)^2$$

The coefficient v can be viewed as penalty term for the cardinality constraint.

An alternative look at $(kQKP^{v^*,u^*}_{DWR(1)})$

- Solving directly

$$
\begin{aligned}
\max \quad & x_p^\top Q x_p + L^\top x_p - v\left(\sum_{j=1}^{n} x_j - k\right)^2 \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \leq b \\
& x_j \in \{0,1\} \qquad\qquad\qquad\qquad j = 1,\ldots,n
\end{aligned}
$$

is computationally non convenient.

An alternative look at $(kQKP^{v^*,u^*}_{DWR(1)})$

- Solving directly

$$
\begin{aligned}
\max \quad & x_p^\top Q x_p + L^\top x_p - v\left(\sum_{j=1}^{n} x_j - k\right)^2 + \tau^{*\top} x + \beta^* \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \le b \\
& x_j \in \{0,1\} \qquad\qquad j = 1,\ldots,n
\end{aligned}
$$

  is computationally non convenient.
- The dual variables “lead” the pricing problem and make it significantly easier to solve.

An alternative look at $(kQKP^{v^*,u^*}_{DWR(1)})$

- Solving directly

$$\max \quad x_p^\top Q x_p + L^\top x_p - v\left(\sum_{j=1}^{n} x_j - k\right)^2 + \tau^{*\top} x + \beta^*$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_j x_j \le b$$

$$x_j \in \{0,1\} \qquad j = 1,\dots,n$$

  is computationally non convenient.
- The dual variables "lead" the pricing problem and make it significantly easier to solve.
- IDEA : warm-start with columns heuristically generated ⇒ good dual variables.

			opt	root
		$(kQKP)$	$(kQKP_{M}^{v^*,u^*})$	$(kQKP_{\overline{DWR}(1)}^{v^*,u^*})$
		Time	Time	Time
n	δ			
50	*25*	132.64	1.16	40.13
	50	507.66	1.12	34.72
	75	384.37	0.89	14.65
60	*25*	12.85	1.95	27.93
	50	1089.85	2.37	32.54
	75	430.88	4.87	16.89
70	*25*	987.44	7.41	64.44
	50	2092.01	19.45	72.07
	75	1560.38	21.51	55.08
80	*25*	2324.91	29.38	78.04
	50	2184.51	48.57	66.70
	75	2032.01	88.06	85.52
90	*25*	1438.01	62.14	89.42
	50	2413.05	239.24	102.25
	75	2986.10	1024.02	491.36
100	*25*	2201.38	234.21	145.81
	50	3157.54	1271.16	289.12
	75	3246.99	1419.84	656.70
Avg		1621.25	248.74	131.30

TABLE : $(kQKP_{\overline{DWR}(1)}^{v^*,u^*})$ root node vs CPLEX

Plan

Conclusion and perspectives

Conclusion

- Dantzig-Wolfe Reformulation and convexification approaches can be used together.
- No dominance between $(kQKP_{DWR})$ and $(kQKP_{\overline{DWR}})$.
- DWR is crucial to obtain strong dual bounds.
- We are able to solve instances of (kQKP) faster than Cplex and Cplex+MQCR.

Perspectives

- Apply this approach to other 0-1 quadratic problems.
- Investigate more in deep how the convexity of the objective function affects the performance of DWR.
- Derive an exact method combining convexification and Branch & Price.

THANK YOU !